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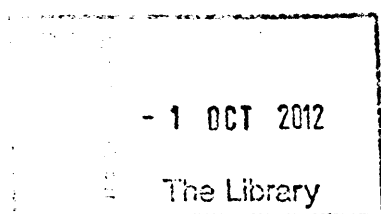
Graph Theory in America 1876 to 1950

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A thesis submitted for the degree of Doctor of Philosophy (PhD) in the Faculty of Mathematics, Computing and Technology of The Open University

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Abstract

This narrative is a history of the contributions made to graph theory in the United States of America by American mathematicians and others who supported the growth of scholarship in that country, between the years 1876 and 1950.

The beginning of this period coincided with the opening of the first research university in the United States of America, The Johns Hopkins University (although undergraduates were also taught), providing the facilities and impetus for the development of new ideas. The hiring, from England, of one of the foremost mathematicians of the time provided the necessary motivation for research and development for a new generation of American scholars. In addition, it was at this time that home-grown research mathematicians were first coming to prominence.

At the beginning of the twentieth century European interest in graph theory, and to some extent the four-colour problem, began to wane. Over three decades, American mathematicians took up this field of study — notably, Oswald Veblen, George Birkhoff, Philip Franklin, and Hassler Whitney. It is necessary to stress that these four mathematicians and all the other scholars mentioned in this history were not just graph theorists but worked in many other disciplines. Indeed, they not only made significant contributions to diverse fields but, in some cases, they created those fields themselves and set the standards for others to follow. Moreover, whilst they made considerable contributions to graph theory in general, two of them developed important ideas in connection with the four-colour problem. Grounded in a paper by Alfred Bray Kempe that was notorious for its fallacious ‘proof’ of the four-colour theorem, these ideas were the concepts of an *unavoidable set* and a *reducible configuration*.

To place the story of these scholars within the history of mathematics, America, and graph theory, brief accounts are presented of the early years of graph theory, the early years of mathematics and graph theory in the USA, and the effects of the founding of the first

institute for postgraduate study in America. Additionally, information has been included on other influences by such global events as the two world wars, the depression, the influx of European scholars into the United States of America, mainly during the 1930s, and the parallel development of graph theory in Europe.

Until the end of the nineteenth century, graph theory had been almost entirely the prerogative of European mathematicians. Perhaps the first work in graph theory carried out in America was by Charles Sanders Peirce, arguably America's greatest logician and philosopher at the time. In the 1860s, he studied the four-colour conjecture and claimed to have written at least two papers on the subject during that decade, but unfortunately neither of these has survived. William Edward Story entered the field in 1879, with unfortunate consequences, but it was not until 1897 that an American mathematician presented a lecture on the subject, albeit only to have the paper disappear. Paul Wernicke presented a lecture on the four-colour problem to the American Mathematician Society, but again the paper has not survived. However, his 1904 paper has survived and added to the story of graph theory, and particularly the four-colour conjecture.

The year 1912 saw the real beginning of American graph theory with Veblen and Birkhoff publishing major contributions to the subject. It was around this time that European mathematicians appeared to lose interest in graph theory. In the period 1912 to 1950 much of the progress made in the subject was from America and by 1950 not only had the United States of America become the foremost country for mathematics, it was the leading centre for graph theory.

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It is my pleasure to express my thanks to the people who helped in the research and writing of this thesis.

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Carrying out research and producing a thesis is often a lonely experience with visits to and correspondence with libraries and the interface with the World Wide Web. However, I would like to thank all the librarians, and particularly Rachel Childs at the Bodleian Library, who have given me assistance; additionally I am grateful for the internet.

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Chapter 1

The early years of American mathematics

Just sixteen years after the Mayflower had landed at Plymouth on the east coast of America, pioneers of North American colonisation founded the first establishment of higher learning in America, at Cambridge in Massachusetts; it was to be another 57 years before the second college was opened. The educational fare offered by the early colleges was simply to transmit known knowledge. Although advances were made over the years, it was not until 1862, when the government intervened with the Morrill Act (see below), that significant steps were made in the education of undergraduates; and in 1876, with the opening of The Johns Hopkins University, similar advances were made in the education of graduates; although the University also taught undergraduates.

During the second half of the nineteenth century, when efforts were being made to provide graduate education opportunities and initiate research in universities and colleges, two men were notable for their contributions; they were Benjamin Peirce and Eliakim Hastings Moore.

Further information on the early colleges in America can be found on their individual web sites [1], [2], [3], [4], [5]; additionally the web site [6] has details of colleges and mathematicians.

1.1 Some early colleges

The first institution of higher education to be established in the American colonies was Harvard College in Cambridge, Massachusetts, in 1636. It was created by the Great and General Court of the Massachusetts Bay Colony. The College was named after a Puritan minister, John Harvard of Charleston, West Virginia, who bequeathed his library and half his estate to it on his death in 1638. John Harvard was born in 1607 in London, England,

and was baptised the same year at St. Saviour's Church (now Southwark Cathedral). He received his MA from Cambridge University in 1635.

Over the following century or so, other universities and colleges were founded. All of these institutions were privately funded and their aim was, essentially, to fit their students for careers in theology, law, medicine and teaching. After Harvard came the College of William and Mary at Williamsburg, Virginia, in 1693; Yale University at New Haven, Connecticut, in 1701; the University of Pennsylvania at Philadelphia in 1740; Princeton University in New Jersey in 1746; Columbia University in New York City in 1754; Brown University, at Providence, Rhode Island, in 1764; and Dartmouth College at Lebanon, New Hampshire, in 1769. All of these institutions were located in the original thirteen states of the union, and increasingly more were to be founded as time and social development continued.

These early colleges did not include a knowledge of mathematics among their entrance requirements, and later when it was considered a prerequisite — at Yale in 1745, Princeton in 1760, and Harvard in 1807 — it was limited to elementary arithmetic only. In 1816, Harvard obliged its applicants to have a greater understanding of arithmetic, and they added algebra in 1820. It was not until after the Civil War of 1861–65 that those other colleges insisted on algebra.

There was little enthusiasm for mathematics in the early years of the American colleges, as demonstrated by the low level of entrance criteria. Members of staff not otherwise usefully employed carried out the teaching of any mathematics that was considered necessary. In 1711, the Reverend Tanaquil Lefevre, the son of a French diplomat, was the first person in the Colonies to be appointed a professor of mathematics, at the College of William and Mary. Isaac Greenwood took up a similar position at Harvard in 1726, but neither held his tenure very long. By 1729, the colonies could boast six professors of mathematics, usually coupled with another subject such as natural

philosophy (physics) or astronomy; all of them were graduates of British universities: Oxford, Cambridge and Edinburgh.

Like all developing nations, America put great store in the education of its population, but the rate of progress, as always, was governed by economics and the calibre of the people available for teaching. Progress was made over the next century in all areas of learning, not least through the increase in the number of institutes of higher education, with the accompanying considerable growth in the importance of mathematics.

The ending of the American Civil War initiated a considerable increase in the amount of disposable money available, both to the government and to the general populace. This new-found affluence, coupled to the Morrill Act, which was authorised by President Abraham Lincoln in 1862, allowed a significant increase in the number of institutions of higher learning in the USA. The Morrill Act provided a fundamental change in the way that higher learning was perceived and funded. It allowed for the distribution of public land to the states and territories for the building of colleges, together with the resources necessary for the teaching, initially, of agriculture and mechanical subjects. This act was conceived as a vehicle to promote and enhance the practical education of the growing industrial population. As well as providing the land and funds to build and run the colleges and universities, it set about making higher education available to more sections of the population than hitherto, with the express intention that this should be for those people who could benefit from further education, irrespective of their financial background. This also had the effect of considerably increasing the admission of women into institutions of higher learning and providing colleges specifically for women and for African Americans, as the former slaves and their descendents are now known.

In the history of graph theory, the most significant of these early American colleges were Harvard and Princeton. Later colleges important to this thesis are the Massachusetts Institute of Technology, founded in 1861 in Cambridge, Massachusetts; The Johns

Hopkins University, founded in 1876 in Baltimore, Maryland; and The Institute of Advanced Study, located at Princeton, established in 1930.

Harvard University

Harvard remains to this day one of the most prestigious places of learning in America and can claim, as graduates, eight presidents of the USA and more than forty Nobel Laureates. Opening with just nine students and a single master, the education provided there was initially based on the European pattern, and the College was modelled on traditional English universities with their classic academic courses, but tailored to the Puritan philosophy of the first colonists. Although the College was never officially associated with any religious denomination, many of its early graduates entered the Puritan Church, taking up positions as clergymen throughout New England.

The first non-clergyman to become President of Harvard College was John Leverett in 1708, and under his and his successors' guidance over most of the eighteenth century, the offered curriculum was widened, particularly in the sciences. Indeed the development was so successful that in 1780 the Massachusetts Constitution officially recognised Harvard as a University.

The formal teaching of mathematics played little part for almost the first two hundred years of the University. Although there had briefly been a professor of mathematics in the 1720s, it was not until much later that recognisable courses in mathematics became available.

The University continued to expand and develop during the early 1800s, acquiring a growing reputation. This was particularly true of the Department of Mathematics and Natural Philosophy under the direction of John Farrar from 1806 to 1836 and then, from 1836 to 1880, the outstanding Benjamin Peirce, unquestionably one of the two leading

American mathematical astronomers of the nineteenth century (Peirce features in detail later in this chapter).

In 1847, the Lawrence Scientific School opened at Harvard, allowing Peirce the opportunity to develop a graduate programme. Farrar made translations of eighteenth-century French works on mathematics, physics and astronomy as aids to improving the level of undergraduate instruction. His resignation due to poor health allowed Peirce to take the lead role in teaching these subjects at Harvard. From 1836 to the end of the century, mathematics at the university was dominated by Peirce and his students. During the early 1830s he produced a number of new sets of course notes, resulting in a further improvement in the quality of teaching.

The Lawrence Scientific School was an early attempt to foster graduate education in individual sciences, in place of the previously labelled natural philosophy or natural history. During the period 1845 to 1865, Harvard was the foremost mathematical research centre in America, with most of the credit going to Peirce. However, it was not until 1912 that Harvard invested in a dedicated postgraduate programme of mathematical research.

Princeton University and the Institute for Advanced Study

Princeton was chartered in 1746 under its original name of The College of New Jersey, by which it was known for its first 150 years. Initially located in Elizabeth, New Jersey, it moved after a year to Newark, and then to Nassau Hall, Princeton, in 1756. Nassau Hall was one of the largest buildings in the new colonies and was named after King William III, Prince of Orange of the House of Nassau. The charter, dated 22 October 1746, was issued by the Province of New Jersey in the name of King George II and stated that ‘any Person of any religious Denomination whatsoever’ may attend. The College was North America’s fourth establishment of higher learning.

The first student intake comprised ten young men, who attended classes in the parlour of the Reverend Jonathan Dickinson. In 1780, Princeton's charter was amended so that the trustees no longer needed to swear allegiance to the King of England. In 1783, the Continental Congress met in Nassau Hall, making Princeton the capital of the newly emerging nation for a short time. Nine Princeton alumni attended the Constitution Convention in 1787, more than from any other American or British institution.

By 1896, the College had developed a sufficiently enhanced educational programme that it was granted university status, was renamed Princeton University after its host city, and in 1900 opened a graduate school, with mathematics research under the guidance of Henry Burchard Fine (1858–1928). Fine published a number of mathematical research papers on numerical analysis and geometry, but was foremost a writer of textbooks. In addition he was a gifted leader, with skills in administration and the development of academic organisations. In the late 1880s, he was an active supporter of the founding of the New York Mathematical Society, which became the American Mathematical Society in 1894.

By this time, Princeton, like most of the American universities, had made little contribution to original mathematics. However, the graduate school brought some success in research for individual members of the mathematics faculty, but the emphasis at Princeton remained that of teaching; research was the poor relation, with little funding and scant facilities. One of the mathematical researchers of that time, Solomon Lefschetz (1884–1972), recalls [7]:

When I came in 1924 there were only seven men there engaged in mathematical research. These were Fine, Eisenhart, Veblen, Wedderburn, Alexander, Einar Hille, and myself. In the beginning we had no quarters. Everyone worked at home. Two rooms in Palmer [Laboratory of Physics] had been assigned to us. One was used as a library, and the other for everything else! Only three members of the department had offices. Fine and Eisenhart [as administrators] had offices in Nassau Hall, and Veblen had an office in Palmer.

In the early 1900s, Fine was the foremost researcher in mathematics at Princeton. In order to increase the amount of research at the newly accredited university he appointed a number of young scholars to preceptorships; one of these was Oswald Veblen in 1905 and another was G. D. Birkhoff in 1908 (see Chapter 4).

Although facilities were less than ideal, much was done to improve the situation during the period 1924 to 1930. In 1926 Fine documented criteria for an enhanced research facility and, with support from others, set about raising the necessary funds. These aims were realised in 1930 with the foundation at Princeton of the Institute for Advanced Study, with mathematics as its first field of study and Albert Einstein among its first mathematicians. This also had the effect of encouraging mathematical research in the University with improvements in its facilities.

The Massachusetts Institute of Technology (MIT)

On 10 April 1861, a charter was approved to found a school of higher education in Boston, Massachusetts. It read ‘An Act to Incorporate the Massachusetts Institute of Technology, and to Grant Aid to Said Institute and to the Boston Society of Natural History . . .’, and four years later it opened to the first students. The efforts of the Institute’s founder and first president, William Barton Rogers, to raise funds were made more difficult due to the outbreak of the American Civil War. As a result, classes were initially held in rented accommodation in the Mercantile Building in Boston. The Institute’s first owned buildings, completed in 1866, were located in Boston’s Back Bay.

During the early years of this (essentially) engineering school, the head of mathematics was John Daniel Runkle, who had been a pupil and protégé of Benjamin Peirce. Runkle had attended the Lawrence Scientific School at Harvard, from which he graduated in 1851, and subsequently worked at the Nautical Almanac Office in Cambridge; in both places, he enjoyed the influence and encouragement of Peirce. Runkle

was Rogers' right-hand man, and they were both influential in planning and defining the Institute's teaching programme. Runkle believed that his department was there to provide mathematical instruction for the engineering students.

On Runkle's death in 1902, his successor, Harry Walter Tyler, set about making the teaching of mathematics a serious subject in its own right, and not just as a service to engineering. In this, he was encouraged by the President, Richard Maclaurin, who also supported Tyler's efforts to expand the department and to promote mathematical research. The Institute relocated to its present site in Cambridge in 1916, housed in what is known as the Maclaurin building, after the president who oversaw its construction.

By the early 1920s, the Massachusetts Institute of Technology had established itself in the top division of mathematics research. It earned a reputation for invention, and many successful and significant companies were founded by the Institution and by its graduates. Over sixty current or former members of the Institute have received Nobel Prizes.

The Johns Hopkins University

The Johns Hopkins University, founded in Baltimore, Maryland, in 1876, was the first research university in the USA, with funds provided through a bequest of \$3,500,000 from the American financier Johns Hopkins. Additionally, he left an equal amount to fund the building of the Johns Hopkins Hospital, which opened in 1893 and provided facilities for training students of the University's medical school. The University opened on 22 February 1876 with Daniel Coit Gilman as its first president. In his installation address, he asked:

What are we aiming at? The encouragement of research . . . and the advancement of individual scholars, who by their excellence will advance the sciences they pursue and the society where they dwell.

With the freedom of starting from nothing, and without the need to change entrenched ideas, Gilman set out to create an academic establishment new to the USA. His guiding

premise was to build a research school that, through scholarship, would improve the individual student's knowledge and enhance the general level of human understanding. He succeeded in developing an atmosphere where teaching and research went hand in hand, and where all faculty members became confident to do both. What Gilman achieved at Johns Hopkins marked a major turning point in the higher education system in the USA and set a challenge to other American colleges and universities.

To implement his aspirations in the mathematics department, Gilman travelled to Europe in 1876 to attract a world-renowned mathematician with a reputation for original research and for encouraging others in research. With the assistance of Benjamin Peirce, he secured the services of the British mathematician James Joseph Sylvester, who was 61 years of age and retired but still young enough in mind to be able to instil enthusiasm in young scholars. The story of the development of mathematics at Johns Hopkins with Sylvester continues in Chapter 2.

1.2 Early mathematics education

The teaching in colleges in the early years of the USA was elementary and followed eighteenth-century English practice, comprising Latin, Greek, philosophy, basic Newtonian mechanics and a little mathematics, including Euclid's *Elements*, the rudiments of trigonometry and basic arithmetic, and some algebra.

A consequence of the 1812–14 War between Britain and the USA over shipping and territory disputes was that many things in America became influenced more by France than by England as hitherto. For mathematics, this meant looking towards a country where mathematics and science were held in respect and benefited from considerable support from government; this attitude had never been a high priority of the British government. This change in emphasis led to the expansion of mathematics and science faculties within

American colleges, and the corresponding creation of additional chairs within these disciplines.

During these early years, little research was carried out within the higher education system in the USA. Although many people working within the system considered research to be prestigious, few facilities were available and there were no internal structures for fostering it. Additionally, because higher education in America was almost exclusively devoted to undergraduate teaching, there was little experience and ability available to develop postgraduate study. It was accepted at the time that promising graduates should travel to Europe, mostly to Germany, for doctoral study and research.

The founding of The Johns Hopkins University initiated a process of change. Its first president, Daniel Gilman, was unlike the presidents of long-established colleges and universities such as Harvard, Princeton and Yale, where they were steeped in entrenched tradition. He recognised that American higher education was far behind that of many European countries. Additionally, he felt that for his new university to survive and grow, it needed to offer an alternative programme to that of extant institutions. As a result, Johns Hopkins placed equal emphasis on undergraduate studies and graduate education incorporating research and support for technical publication. One of Gilman's objects was to make the United States of America competitive with Europe, and to help achieve this object he set about recruiting internationally respected scholars who had a long history of research and of encouraging research in others. It was at Johns Hopkins that the *American Journal of Mathematics* was founded in 1878 (see Chapter 2).

1.3 Two scholars

Two American mathematicians who made considerable contributions to the development of teaching and graduate study in the USA, and who in differing ways were significant in

the history of what was to become graph theory, were Benjamin Peirce of Harvard University and Eliakim Hastings Moore of the University of Chicago.

Benjamin Peirce (1809–1880)

Throughout the relatively short history of American scholarship, and mathematics in particular, there have been dynasties, albeit mostly of two-generation duration. One of these was the Peirce family, whose head was Benjamin Peirce (pronounced *purse*). His offspring included his highly acclaimed son Charles Sanders Peirce, mathematician, logician and philosopher (see Chapter 2).

After graduating from Harvard in 1829, Peirce taught for two years at George Bancroft's Round Hill School in Northampton, Massachusetts, before returning to Harvard and joining the mathematics faculty where he remained for the rest of his life.

Peirce was perhaps the first American-born professor of mathematics who considered research to be part of his role, and not just something to do in his spare time. During his time at Harvard, he was influential in elevating the status of the university to that of a leading national institution. He is regarded as having made a considerable contribution to the emergence of mathematical research in the USA. In 1847, the Lawrence Scientific School was founded at Harvard, which allowed Peirce to teach graduates and encourage them in research. The programme he developed was so advanced that he averaged only two students per year.

Peirce's research topics included celestial mechanics, applications of plane and spherical trigonometry to navigation, geodesy, linear associative algebra, number theory and statistics. In particular, he published, at his own expense, *Linear Associative Algebra* in 1870. This work classified all complex associative algebras of dimension less than 7, and laid down the foundations of a general theory of linear associative algebras. In addition, he calculated multiplication tables for more than 150 new algebras. This work

was highly influenced by the study of quaternions by William Rowan Hamilton (1805–1865) (see Appendix I) who was the leading mathematician and astronomer in Ireland during the 19th century. Peirce's work was considered the first American treatise on modern abstract algebra and the first important research to come out of the USA in the area of pure mathematics.

It has been said that Benjamin Peirce made three significant contributions to mathematics: as a stimulating and inspiring teacher and dedicated researcher, as an outstanding scholar in the field of statistics, and for his son C. S. Peirce [8].

Eliakim Hastings Moore (1862–1932)

One of the mathematicians to take advantage of the home-grown postgraduate training, as well as benefit from overseas study, was Eliakim Hastings Moore. He graduated from Yale in 1883 and received his doctorate in 1885, before spending the academic year 1885–86 in Germany where he attended the Universities of Göttingen and Berlin. On his return to the USA, he was a high-school instructor for a year before becoming a tutor at Yale for two years. In 1889, Moore joined Northwestern University and then, in 1892, moved to the new University of Chicago as Professor of Mathematics and acting head of the Department of Mathematics. Chicago University, founded in 1890 largely with money provided by the founder of Standard Oil, John D. Rockefeller, had as its main objective the development of postgraduate study and research, together with undergraduate instruction.

It was Chicago's first president, the outstanding administrator William Rainey Harper, who recruited Moore. Together they developed an excellent mathematics faculty. Chicago's doctoral graduates went on to establish and expand many important departments of mathematics across America over the first decades of the twentieth century. Indeed, although Moore made no direct contribution to graph theory, two of his postgraduate students were to publish major works on the subject.

Moore was to stay at Chicago for 40 years, where he devoted considerable time to the building of the mathematical community in America. In 1893 he was instrumental in organising the *World's Columbian Exposition* in Chicago to commemorate the 400th anniversary of the discovery of the Americas. This attracted attendance and contributions from 45 mathematicians from Austria, Germany, Italy and 19 states of the Union, as well as papers from French, Russian, and Swiss scholars, and was the first international mathematical meeting to be held in the USA. He then encouraged the New York Mathematical Society to expand its boundaries and change its name to the American Mathematical Society in 1894, and went further by setting up a Chicago section of the Society in 1897. Next, he energetically lobbied the Society to produce a learned periodical to publish research papers which should be predominantly by American mathematicians. The Society agreed to do this, and in 1899 there appeared the first issue of the *Transactions of the American Mathematical Society*, with Moore as Editor-in-Chief. In 1906, Moore presented the Society's Colloquium lectures.

Thirty-one research students earned their PhDs under Moore's supervision, and by the end of the twentieth century he was recorded as having had over 9500 doctoral descendants. A number of people have commented that this list of descendants reads like a *Who's Who* of twentieth-century American mathematicians. Two of his successful students, Oswald Veblen and George David Birkhoff, went on to become leading American mathematicians in the first half of the twentieth century, making significant contributions to graph theory; they form the subject of Chapter 4.

Conclusion

As with many things in the USA during the nineteenth and twentieth centuries, the development of higher education moved apace, not least in mathematics. By 1910, and through to the outbreak of World War II, Harvard, Princeton, Chicago, and Johns Hopkins

were not only the leading mathematical establishments in America, but were comparable to many of the top European universities. This was due, in large part, to the lead set by Johns Hopkins, and the significant change in graduate education at Harvard in the early 1900s, and soon followed by other universities. Other institutions that followed Johns Hopkins' lead included Princeton and Yale, as well as newly formed institutions such as Clark University and the University of Chicago. This changed the principal emphasis from mathematical education to mathematical research. Moreover, the seminal work done by Peirce and Moore in developing postgraduate study in America proved to be of immense importance for the future of mathematics.

The story of mathematical research, the influence of Johns Hopkins and Sylvester, the initial interest in graph theory in America, and some of its nineteenth-century pioneers are explored in the next chapter.

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Chapter 2

The beginnings of American graph theory

As has been seen, The Johns Hopkins University was the first establishment of education in the USA that had been founded partly with the aim of encouraging and providing facilities for research. The University's first president set out to employ the very best scholars to head its departments. Mathematics was the first faculty to open, with James Joseph Sylvester as its guiding light. Sylvester published several technical papers, including a few relating to graph theory.

The story of Johns Hopkins and its mathematics during its first few years is essentially that of Sylvester, but also involves contributions from other notable figures. Alfred Bray Kempe, a fellow Briton, whilst not directly involved in the development of graph theory in America, is important to the story, because of the famous error in his solution of the four-colour problem; this was to have an unwitting influence on events on graph theory in the USA and on the search for a valid solution.

However, there were two other scholars important to the early history of Johns Hopkins and to the development of mathematics in the USA. The first was William Edward Story, a mathematician with a talent for organisation but little luck. The second was Charles Sanders Peirce, a brilliant but somewhat wayward polymath.

Further information on Sylvester can be found in [1], [2].

2.1 James Joseph Sylvester (1814–1897)

Sylvester was a mathematician, former actuary and barrister, who had enjoyed an eventful career in his native country, but was already retired when he received Daniel Gilman's invitation to become the first Professor of Mathematics at Johns Hopkins in 1876. Although many considered Sylvester the finest mathematician in the English-speaking

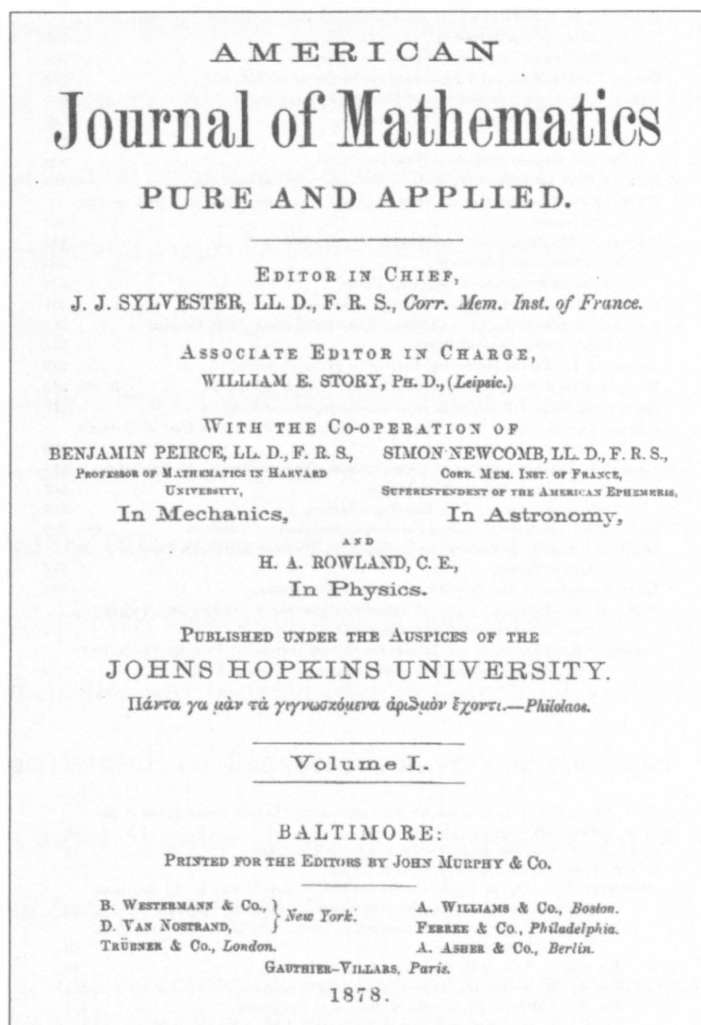
world, he was surprised and pleased by the invitation. Benjamin Peirce, a friend of Sylvester, had written to Gilman to urge him to engage Sylvester. Peirce's letter of 18 September 1875 included the following passage [3]:

Hearing that you are in England, I take the liberty to write you concerning an appointment in your new university, which I think would be greatly for the benefit of our country and of American science if you could make it. It is that of one of the two greatest geometers of England, J. J. Sylvester. If you enquire about him, you will hear his genius universally recognised but his power of teaching will probably be said to be deficient. Now there is no man living who is more luminary in his language, to those who have the capacity to comprehend him than Sylvester, provided the hearer is in a lucid interval. But as the barn yard fowl cannot understand the flight of the eagle, so it is the eaglet only who will be nourished by his instruction . . . Among your pupils, sooner or later, there must be one, who has a genius for geometry. He will be Sylvester's special pupil — the one pupil who will derive from his master, knowledge and enthusiasm — and that one pupil will give more reputation to your institution than ten thousand, who will complain of the obscurity of Sylvester, and for whom you will provide another class of teachers . . . I hope that you will find it in your heart to do for Sylvester — what his own country has failed to do — place him where he belongs — and the time will come, when all the world will applaud the wisdom of your selection.

On taking up his appointment in May 1876, at a salary of \$5000 per annum (paid in gold) [4], Sylvester set about realising Gilman's objective, by initiating research work in the mathematics department. He selected two graduate fellows, George Bruce Halsted and Thomas Craig, to join the mathematics faculty, and in the autumn William Story was recruited from Harvard.

Sylvester presented his inaugural lecture on 22 February 1877, the day of the celebrations of the first anniversary of the official opening of the University. His speech covered many subjects, including how mathematics should be taught and studied, the role of Johns Hopkins in the development of mathematics in the USA, and indeed that of further education in America. He also included an attack on English universities that discriminated against all who were not Protestant Christians. As a sufferer of this prejudice

himself, he took the opportunity to acclaim the damage done to the furtherance of higher education in England by the exclusion of Jews, Catholics, and others.



Frontispiece of the first issue of the American Journal of Mathematics.

Sylvester is credited as the founder of the *American Journal of Mathematics*, which is still in publication today, and with the help of Story he published the first issue in 1878. It was intended that the *Journal* be a vehicle for dialogue between American mathematicians, although space was also made available for foreign contributions. As can be seen from the index overleaf the first issue contained contributions from the British scholars Arthur Cayley, W K Clifford, Edward Frankland (although his initial is given as A) and, of course, Sylvester; the Americans Simon Newcomb, C S Peirce and Story; plus papers from Sylvester's students T Craig, F Franklin and G B Halsted.

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Index of the first issue of the American Journal of Mathematics.

The first six volumes of the Journal, which covered the years 1878–1884 and for which Sylvester was responsible, contained nearly 200 papers. Sylvester was included in each of the volumes with a total of 32 entries and Cayley contributed to five volumes. Cayley (1821–1895) was an English mathematician who met Sylvester when they were both studying law in London, becoming lifelong friends and collaborators on mathematical

matters. In addition to the British mathematicians included in Volume I other Europeans who featured in Volumes II–VI were Julius Petersen (see Chapter 3) and Kempe; other Americans having papers in these latter volumes included Benjamin Peirce and all eight of Sylvester’s doctoral students. Sylvester must have initiated a publicity campaign to promote the new publication as its ‘List of Subscribers’ on 1 July 1878 totalled nearly 150, of which 36 were institutions some of whom took multiple copies. Three addresses in Paris were listed, including La Bibliothèque de l’Ecole Polytechnique, and six in England, including the University Library, Cambridge, and two in Canada.

Yale University was the first American institution to confer the PhD degree (in 1861), and in 1862 awarded the PhD degree in mathematics to J H Worrall. In the period 1862–1869 only Yale conferred this degree and only in three cases. In the period 1870–1879 the PhD degree in mathematics was awarded once at Cornell University, once at Dartmouth University, twice at Harvard, on four occasions at Yale and twice at Johns Hopkins. During his time at Johns Hopkins University, Sylvester supervised eight postgraduate students writing their doctoral theses [5]. These were:

1. Thomas Craig, *The representation of one surface upon another, and some points in the theory of the curvature of surfaces*, 1878.
2. George Bruce Halsted, *Basis for a dual logic*, 1879.
3. Fabian Franklin, *Bipunctual coordinates*, 1880.
4. Washington Irving Stringham, *Regular figures in n -dimensional space*, 1880.
5. Oscar Howard Mitchell, *Some theorems in numbers*, 1882.
6. William Pitt Durfee, *Symmetric functions*, 1883.
7. George Stetson Ely, *Bernoulli’s numbers*, 1883.
8. Ellery William Davis, *Parametric representations of curves*, 1884.

By this time, Sylvester and his *Mathematical Seminarium*, as he called his school of mathematics, was being recognised in American mathematical circles and in Europe. Indeed, papers published by this group, most of which appeared in the *American Journal of*

Mathematics, were widely read at home and abroad. George Andrews [6] recently commented that the collective output during these years amounted to a ‘monumental’ contribution to combinatorics, and it was widely accepted that Sylvester and his school were succeeding in putting America on the mathematical map.

In December 1879 the University issued the first of its *Johns Hopkins University Circulars*. This publication was intended initially to communicate throughout the University the full scope of research being undertaken, but it also included correspondence between (and information of) members of the various faculties. In 1883 Sylvester, in a letter to Cayley, wrote [7] that the *Circulars* acted as ‘a sort of record of progress in connection with the work and personality of the Johns Hopkins’. Indeed Sylvester published many notes, papers and some lectures in the *Circulars*.

Sylvester was very happy at Johns Hopkins, not least because, for the first time in his life, he was able to teach and carry out research based on his own ideas and chosen topics within a formal university environment.

Since the mid-1850s Sylvester had been thinking of partition theory on which he corresponded with Cayley. In 1859 he presented a series of public lectures at King’s College, London, covering the work on partitions he had done the previous winter and spring. During his time at Johns Hopkins he further worked on and developed the theory of partitions, a subject that had first been studied in depth by Leonhard Euler (1707–1783). Sylvester’s ideas are still of value today and George Andrews wrote [8]:

The modern combinatorial theory of partitions was founded by Sylvester at Johns Hopkins. Most of the work of Sylvester and his students was gathered together in an omnibus paper entitled *A constructive theory of partitions, arranged in three acts, an interact and an exodion*. The philosophy of the work is perhaps best summarized in the first paragraph of Act I. On Partitions as Entities:

In the new method of partitions it is essential to consider a partition as a definite thing, which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of the parts shall be according to their order of

magnitude. A leading idea of the method is that of correspondence between different complete systems of partitions regularized in the manner aforesaid. The perception of the correspondence is in many cases greatly facilitated by means of a graphical method of representation, which also serves per se as an instrument of transformation.

Sylvester's work, together with that of his students, was assembled in this paper, which was published in two parts, in 1882 and 1884 [9], and was the first major combinatorial study of partitions since Euler. Although the paper was published under Sylvester's name, its contents included individual credits to those parts that had been formulated by his graduate students.

A *partition* of an integer is a representation of it as a sum of positive integers — for example, $5 + 4 + 4 + 2$ is a partition of 15. This can be graphically represented as:

```

•   •   •   •   •
•   •   •   •
•   •   •   •
•   •

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The paper defines the conjugate of a partition as that found by interchanging rows and columns, so that the conjugate of the above graphical representation is:

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•   •   •   •
•   •   •   •
•   •   •
•   •   •
•

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which is the partition $4 + 4 + 3 + 3 + 1$. Furthermore a self-conjugate partition is a partition that is identical with its conjugate — for example, $4 + 3 + 2 + 1 = 10$ is graphically represented as:

• • • •
 • • •
 • •
 •

In this case the changing of rows and columns results in the same partition.

The paper referred to an integer n being partitioned into j parts with i being the largest part. So in the example, $4 + 4 + 3 + 3 + 1$, $n = 15$, $j = 5$ and $i = 4$. The paper went on to develop techniques for determining the number of such partitions.

Like William Kingdon Clifford (1845–1879), a graduate of Trinity College, Cambridge, and potentially one of the major mathematicians of his time before his untimely death, Sylvester believed there to be a direct connection between chemistry and algebra. Sir Edward Frankland (1825–1899) was a British scientist who held appointments in Britain and in continental Europe and was for many years responsible for the continuous analysis of London water supplies; he also served on a Royal Commission on water pollution. In 1866 he published his *Lecture Notes for Chemical Students* [10]. This introductory text began by explaining how atoms and bonds could be graphically depicted by circles and connecting lines. Frankland then went on to list the symbolic and graphic notation for many chemical compounds, beginning with water. Symbolic formulae are expressions of the atoms and their quantities which combine to form chemical compounds — for example, water having the symbolic notation as OH_2 . Graphic notation is where each atom is shown separately, represented by a letter enclosed in a circle and where all single and multiple bonds are identified by lines joining the appropriate circles — for example, water is shown as:



As Sylvester was already convinced of the connection between chemistry and algebra he was very much taken with Frankland's paper. In 1878 Sylvester wrote a short note [11], published in *Nature* and a lengthy paper [12] on the idea. It was in the note that the word *graph* (in the sense of graph theory) was used for the first time and the first two paragraphs of the note show his enthusiasm for the subject and how much he was energised by Frankland's paper:

It may not be wholly without interest to some of the readers of *Nature* to be made acquainted with an analogy that has recently forcibly impressed me between branches of human knowledge apparently so dissimilar as modern chemistry and modern algebra. I have found it of great utility in explaining to non-mathematicians the nature of the investigations which algebraists are at present busily at work upon to make out the so-called *Grundformen* or irreducible forms appurtenant to binary quantics taken singly or in systems, and I have also found that it may be used as an instrument of investigation in purely algebraical inquiries. So much is this the case that I hardly ever take up Dr. Frankland's exceedingly valuable *Notes for chemical students*, which are drawn up exclusively on the basis of Kekulé's exquisite conception of valence, without deriving suggestions for new researches in the theory of algebraical forms. I will confine myself to a statement of the grounds of the analogy, referring those who may feel an interest in the subject and are desirous for further information about it to a memoir which I have written upon it for the new *American Journal of Pure and Applied Mathematics*, the first number of which will appear early in February.

The analogy is between atoms and *binary* quantics exclusively.

The note and the paper were like much of Sylvester's writing: they were not just scholastic but verged on the flowery. The paper that expanded on the note was published in the first volume of the *American Journal of Mathematics* and entitled *On the application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics, — with three appendices*. The first two paragraphs give a flavour of his prose:

By the *new* Atomic Theory I mean that sublime invention of Kekulé which stands to the *old* in a somewhat similar relation as the Astronomy of Kepler to Ptolemy's, or the System of Nature of Darwin to that of Linnaeus, — like the latter it lies outside of the immediate sphere of energetic,

basing its laws on pure relations of form, and like the former as perfected by Newton, these laws admit of exact arithmetic definitions.

Casting about, as I lay awake in bed one night, to discover some means of conveying an intelligible conception of the objects of modern algebra to a mixed society, mainly composed of physicists, chemists and biologists, interspersed only with a few mathematicians, to which I stood engaged to give some account of my recent researches in this subject of my predilection, and impressed as I had long been with a feeling of affinity if not identity of object between the inquiry into compound radicals and the search for “Grundformen” or irreducible invariants, I was agreeably surprised to find, of a sudden, distinctly pictured on my mental retina a chemico-graphical image serving to embody and illustrate the relations of these derived algebraical forms to their primitives and to each other which would perfectly accomplish the object I had in view, as I will now proceed to explain.

Another bizarre example of unlikely phrasing in a technical paper appears later:

Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson. Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other. In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry the idea enveloped in matter, depending in part on manual processes and the resources of art for its due manifestation.

In this paper he again heaps praise on Frankland, saying:

The more I study Dr. Frankland’s wonderfully beautiful little treatise the more deeply I become impressed with the harmony or homology (I might call it, rather than analogy) which exists between the chemical and algebraical theories.

The analogy that Sylvester was trying to make was between binary quantics and atoms. A *binary quantic* is a homogeneous expression of two variables, such as $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, with an invariant being a function of the coefficients a , b , c and d that essentially remains unaltered under linear transformations. He makes it obvious that this idea evolved from the use of diagrammatical representation of chemical compounds. The

note included the following explanation of the connection between atoms and binary quantics:

The analogy is between atoms and binary quantics exclusively.

I compare every binary quantic with a chemical atom. The number of factors (or rays, as they may be regarded by an obvious geometrical interpretation) in a binary quantic is the analogue of the number of *bonds*, or the *valence*, as it is termed, of a chemical atom.

Thus a linear form may be regarded as a monad, a quadratic form as a duad, a cubic as a triad, and so on.

An invariant of a system of binary quantics of various degrees is the analogue of a chemical substance composed of atoms of corresponding valences.

The order of such invariant in each set of coefficients is the same as the number of atoms of the corresponding *valence* in the chemical compound . . . The weight of an invariant is identical with the number of the bonds in the chemicograph of the analogous chemical substance, and the weight of the leading term (or basic differentiant) of a co-variant is the same as the number of bonds in the chemicograph of the analogous compound radical. Every invariant and covariant thus becomes expressible by a graph precisely identical with a Kekuléan diagram or chemicograph . . . I give a rule for the geometrical multiplication of graphs, that is, for constructing a graph to the product of in- or co-variants whose separate graphs are given.

The connection between chemistry and algebra was but one of the more fanciful ideas Sylvester had during his long academic career. Despite the details contained in both the note, the associated paper, and the subsequent correspondence by chemists and mathematicians, his ideas were at the time considered only a passing connection between F. A. Kekulé's notation for chemical compositions and the theory of trees developed by Cayley.

Cayley produced a number of graph theory papers between 1857 and 1889. In 1857, he published the first paper [13] to use the word *tree*, in the graph theory sense, although both Kirchhoff (see Appendix I) and Karl Georg Christian von Staudt (1798–1867) had used the idea around ten years earlier. The term 'tree' as described in [13] 'arose ... from the study of operators in the differential calculus' and was defined as a connected graph

that contains no cycles; it follows that the number of edges is one fewer than the number of vertices and a connected graph with these properties must be a tree. Cayley's paper, inspired by Sylvester's work on what he called 'differential transformation and the reversion of serieses', dealt with *rooted trees* only, in which one particular vertex is denoted as the root.

Cayley published papers where he combined his work on chemical compositions and his studies of trees. In 1874, he presented a paper *On the mathematical theory of isomers* [14]. Isomers are compounds that have the same chemical composition but different atomic configurations. This short paper allowed Cayley to describe how his work on trees could be used in the study isomerism. Two further papers, one in 1875 [15] and the other in 1877 [16], also dealt with the connections between trees and chemical composition. This area of his scholarship generated terminology that is now standard in the field of graph theory.

Sylvester was somewhat apprehensive that his work on the analogy between chemistry and algebra might not meet with universal acceptance. Perhaps he suspected that it was doomed, as he wrote to Newcomb that he felt others might believe it to be 'over fanciful' [17]. Simon Newcomb (1835–1909) was Professor of Mathematics and Astronomer at the Naval Observatory in Washington: he later became Professor of Mathematics and Astronomy at Johns Hopkins University, and was President of the American Mathematical Society from 1897 to 1898. He was also the maternal grandfather of Hassler Whitney (see Chapter 6).

Although there was some academic debate on the theory, it soon ran its course as it became apparent that the only probable link between chemistry and algebra was 'the use of a similar notation' [18]. This was not the only suppositional theory promoted by Sylvester during his long career.

During his time at Johns Hopkins Sylvester published two papers on trees, one in 1879 [19], while the other appeared in the first volume of the *Circulars* that covered the years 1879 to 1882 [20].

During the academic year 1881–82, Sylvester arranged for Cayley to visit Johns Hopkins. While in the USA, Sylvester was deprived of his frequent meetings with Cayley and sent him a number of letters during the early months of 1881 inviting him to teach for a period at Johns Hopkins. Sylvester painted an encouraging picture of the social and academic life in Baltimore and suggested that Cayley would be rewarded academically and financially. Sylvester undoubtedly felt the lack of mathematical peers at Johns Hopkins and in the USA generally, especially after the death of Benjamin Peirce in 1880. His letters, and a visit to Cayley in Cambridge in August 1881, persuaded Cayley to visit Johns Hopkins for six months during the spring semester of 1882, and whilst there to present a series of lectures. Cayley also had papers published in the *Johns Hopkins University Circulars* and the *American Journal of Mathematics*.

In February 1883 Henry Smith, the Oxford Savilian Professor of Geometry, died, thus prompting a search for a successor, preferably an Oxford man. News reached Sylvester and on 16 March he wrote to Cayley indicating that he would probably offer himself as an applicant and in December he was unanimously elected to the chair. Sylvester had submitted his resignation to Johns Hopkins in the autumn and returned to Britain towards the end of the year.

His legacy in the USA was that Johns Hopkins' success in establishing a successful graduate school, which invested time and effort into training future researchers, had an effect on other educational institutions; additionally his legacy included the founding of the *American Journal of Mathematics*. Many other institutions established graduate schools, and the level of mathematical research within America gradually improved, with the result that it was no longer necessary for graduates to journey abroad for postgraduate study,

although some still chose to do so. Sylvester was then well past what we now accept as normal retirement age, and gave an additional reason for leaving Johns Hopkins in a letter to Felix Klein [21]:

. . . because I did not consider that my mathematical erudition was sufficiently extensive nor the vigour of my mental constitution adequate to keep me abreast of the continually advancing tide of mathematical progress to that extent which ought to be expected from one on whom practically rests the responsibility of directing and moulding the mathematical education of 55 million of one of the most intellectual races of men upon the face of the earth.

2.2 William Edward Story (1850–1932)

William Story, who experienced a number of setbacks, was one of the first American mathematicians to study for a degree at a German university, gaining his PhD degree for his thesis, *On the algebraic relations existing between the polars of a binary quantic*, from Leipzig University on 31 July 1875 [22]. He then returned to the USA, becoming a tutor at Harvard University. Story had been known to have made an impression on Benjamin Peirce when he was an undergraduate at Harvard and that impression increased as Story carried out his duties as a tutor. Peirce was sufficiently impressed that when Sylvester approached Peirce for suggestions of suitable mathematicians worthy of consideration to join the newly founded Johns Hopkins Department of Mathematics, Peirce recommended Story. Sylvester decided to return to England for the summer months of 1876 so as to avoid the American heat at that time of year, and so it was left to Gilman to interview Story and make any decision regarding his employment as Sylvester's assistant. Gilman's initial telegraphed approach was not enthusiastically received as Story found it a little patronizing and his reply was perhaps a trifle sharp, but did ask for an interview. During the interview Story outlined his ideas for a learned mathematical journal and a student society. Before accepting the offered position at Johns Hopkins, Story endeavoured to better his status at Harvard but met with no success.

In the autumn of 1876, Story moved to Baltimore as an ‘Associate’ (equivalent to an assistant professor at some other universities); in 1883 the University introduced the title of Associate Professor, and Story was promoted to this position.

Initially things went well for Story. He set about helping to develop the mathematics department, and his preference was to model it on the example he had experienced whilst in Germany. He assisted Sylvester in setting up the *American Journal of Mathematics* and was intimately involved in the founding of a Mathematical Society within the University. Roger Cooke and V Frederick Rickley have said [22]:

There is evidence that Story succeeded in founding his student mathematical society. *The Johns Hopkins University Circulars*, which are a rich source of information about the university, contain titles and reports of the talks given at the monthly meeting of the “Mathematical Society”. From one of these we learn that when Lord Kelvin lectured at Hopkins in 1884, he spoke to a group of mathematicians who called themselves “the coefficients” [Gilman 1906a, p. 75]

As Sylvester was not good with finance or management, he appointed Story as Associate Editor-in-Charge of the *Journal*, and praised his second-in-command [17]:

Story is a most careful managing editor and a most valuable man to the University in all respects and an honor to the University and its teachers from whom he received his initiation.

However, the way that the *Journal* was run caused friction between Story and Sylvester. This was not a personal difference, but a difference in the way that they believed that the journal should be edited. The situation was brought to a head during Sylvester’s absence from America by the publication of Kempe’s famous but flawed paper on the *four-colour problem*, which is reviewed below. The four-colour problem is — can every map drawn on the plane be coloured with at most four colours such that no two neighbouring countries are coloured the same?

During his time at Johns Hopkins, it was Sylvester’s custom to spend each summer in England, leaving the USA in late spring and returning for the start of the next academic year. Story was left in charge for the duration of Sylvester’s annual leave [22].

Cayley also worked on the four-colour problem. On 13 June 1878, he and Kempe were present at a meeting, of the London Mathematical Society where he raised a query that was recorded in the Society's *Proceedings* [23]:

Questions were asked by Prof. Cayley F.R.S. – Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four colours are required, so that no two adjacent counties should be painted in the same colour.

This was repeated in a report of the meeting in *Nature* on 11 July 1878 [24]. These reports were, for many years, believed to be the earliest printed references to the four-colour problem.

In a short note in 1879 [25], Cayley set out to describe succinctly the difficulties inherent in tackling the four-colour problem. The paper included a positive suggestion that when developing a proof, restrictions could be imposed on maps, a portent of things to come. One restriction was that they can be *cubic maps* (those with exactly three countries at each meeting point). He also pointed out that if the four-colour conjecture were true then a map can be constructed so that only three colours are adjacent to the exterior boundary.

Alfred Bray Kempe (1849–1922), a former student of Cayley, was yet another English mathematician who was also a barrister. Most of Kempe's early mathematical work was associated with the application of geometry to mechanical linkages. He is, however, remembered most for his celebrated (but fallacious) proof of the four-colour conjecture. His interest in the topic had been initiated by Cayley's query to the London Mathematical Society [23] and Cayley's memoir [25] in the *Proceedings of the Royal Geographical Society* in April 1879. Shortly afterwards, on 17 July 1879, Kempe announced a 'solution' in *Nature* [26]. On 26 February 1880 Kempe's 'simplified' versions were published, one an untitled abstract [27], in the *Proceedings of the London Mathematical Society* and the second in *Nature*, under the title *How to colour a map with four colours* [28].

Kempe's paper was entitled *On the geographical problem of the four colours* [29] and it is in this work that he claimed to have solved the four-colour conjecture. He explained it thus:

Some inkling of the nature of the difficulty of the question, unless its weak point be discovered and attacked, may be derived from the fact that a very small alteration in one part of a map may render it necessary to recolor it throughout. After a somewhat arduous search, I have succeeded, suddenly, as might be expected, in hitting upon the weak point, which proved an easy one to attack. The result is, that the experience of the map-makers has not deceived them, the maps they had to deal with, viz: those drawn on simple connected surfaces, can, in every case, be painted with four colours. How this can be done I will endeavour – at the request of the Editor-in-Chief – to explain.

The Editor-in-Chief was the journal's founder, J. J. Sylvester.

Unfortunately, Kempe's paper contained a fatal error, which was uncovered eleven years later, during which time his proof was generally accepted. Percy John Heawood (see Chapter 3), had heard of Kempe's paper from Henry Smith, Oxford's Savilian Professor of Geometry at Oxford University, who found the error in 1890. Kempe's complete paper was published later in the year in Volume 2 of the *American Journal of Mathematics* and led to a falling out between Sylvester and Story.

2.3 Kempe's paper [29]

In the first part of his paper, Kempe created a version of Euler's formula applicable to maps, from which he developed the formula:

$$5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - \text{etc.} = 0,$$

where, for each k , d_k denotes the number of districts of the map with k boundaries. Since only the first five terms are non-negative, not all of d_1 to d_5 can be zero. Kempe noted:

... every map drawn on a simply connected surface must have a district with less than six boundaries.

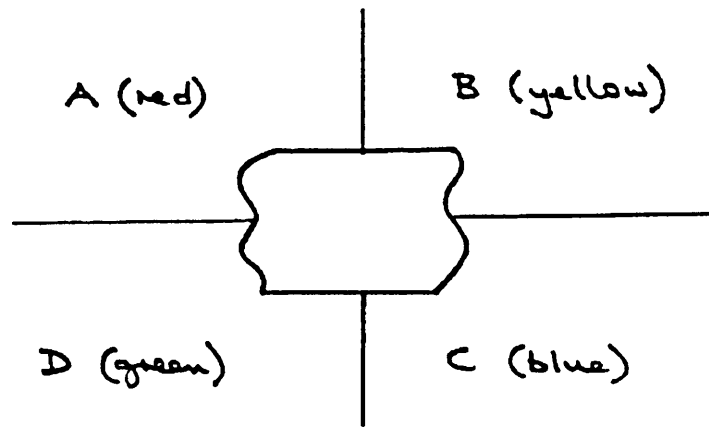
Using this result, Kempe went on to develop an algorithm for colouring any map, using a system of patches. This process involved selecting a district with five or fewer neighbours,

and then covering it with a blank piece of paper – a patch – a little bigger. He then joined all the boundaries that touch the edge of the patch to a single point on the patch; this has the effect of reducing the number of districts by 1. The process is then repeated until only one district remains; as Kempe put it, ‘The whole map is patched out’. This remaining district is then coloured with any of the four colours.

He then reversed the patching process, taking off one patch at a time and successively colouring the uncovered districts with any of the four colours available, until the original map was coloured with four colours. Unfortunately, his explanation of this step failed to provide a rigorous argument. This patching procedure works provided that each restored district has at most three boundary lines. However if a district has four or five boundary lines, then it may be surrounded by districts using all four colours. To overcome the difficulty, Kempe developed a strategy now called the *method of Kempe-chains*; a method of colouring maps or planar graphs in which two colours are interchanged so that regions that previously could not be coloured properly be coloured. Although the paper did not provide the proof of solution to the four-colour problem as it claimed, this important line of argument became one of the standard tools for tackling the colouring of maps and other colouring problems. It was in the incorrect application of the method of Kempe-chains when recolouring a map that contains a five-sided district that gave rise to his famous *faux pas*.

His argument was that, given a map where all districts except one are coloured and assuming that the coloured districts surround the uncoloured district and are assigned all of the four available colours then his method could be applied. (If fewer than four colours are used, the uncoloured district can be assigned one of the unused colours without further development of the argument).

As an example, the following map has an uncoloured district surrounded by four coloured districts.



The districts assigned the colours *red* and *blue* can either be connected by a continuous chain of *red* and *blue* districts or not so connected. In the latter case, it is permissible to exchange the colours *red* and *blue* in the chain of *red-blue* districts connected to district *A* without altering the colour of *C*; this results in both districts *A* and *C* being coloured *blue*, so that the uncoloured district can be coloured *red*. However, if a continuous chain of *red-blue* districts joins *A* and *C* then there would be no advantage in making such an interchange of colours. In this case, it follows that no continuous chain of *yellow-green* districts can join *B* and *D*. Therefore, either of the *yellow-green* chains connected to *B* or *D* can be recoloured, making *B* and *D* either both *yellow* or both *green*. This procedure allows the four districts surrounding the uncoloured one to be coloured with three colours, leaving the fourth colour for the centre district.

The mistake that Kempe made was to make two colour interchanges at the same time, without realising that in so doing the result would be that some adjacent districts would be the same colour. He was considering the colouring of a map containing a district with five sides, when he made two simultaneous recolourings on strings of districts; either recolouring by itself would have been valid.

Kempe's two further papers on this subject in the following two years, although they were intended to be improved versions of his 'proof', both contained the fundamental error of the original paper. The first, an untitled abstract [27], was published in the *Proceedings*

of the *London Mathematical Society* in 1879, where Kempe states that he has presented a proof of the four colour conjecture in the *American Journal of Mathematics* and that his abstract is ‘simpler, and is free from some errors which appeared in the former’. The paper does indeed provide a simpler description of his reduction and patching method and includes instructions for interchanging colours within chains, but it does not indicate a recognition of his fatal error.

The second follow-up paper, *How to colour a map with four colours* [28], was published in *Nature* in 1880 and was similar in content to the untitled abstract. It was again offered as a simplification, and in Kempe’s own words, ‘I have succeeded in obtaining the following simple solution in which mathematical formulae are conspicuous by their absence’. On reading these two papers today, one cannot help arriving at the conclusion that Kempe was by no means trumpeting aloud his claimed achievement, but was modestly confident that he had found the solution to a problem that had vexed and entertained a considerable number of mathematicians, both professional and amateur.

2.4 Story’s note

Story had reviewed Kempe’s paper and on 5 November 1879 presented the salient points of the ‘proof’ to an audience of 18 at a meeting of the Johns Hopkins Scientific Association. After presenting Kempe’s paper, Story offered ‘a number of minor improvements’ which he put in the form of a note that was intended ‘to make the proof absolutely rigorous’. Story’s *Note on the preceding paper* [30] published in the *American Journal of Mathematics*, immediately followed Kempe’s paper. In his note, Story addressed special cases that Kempe had not covered in his paper. He used both the patch method and Euler’s formula, as Kempe had done, but endeavoured to be more precise in the use of the formula for various examples contained in Kempe’s paper. It is unfortunate that Story was not able to identify the major flaw in Kempe’s ‘proof’ in his review of

Kempe's paper and in the development of his own contribution. Story's opening paragraph set out his intention, saying:

... it seems desirable, to make the proof absolutely rigorous, that certain cases which are liable to occur, and whose occurrence will render a change in the formulae, as well as some modification of the method of proof, necessary, should be considered separately ...

Story concentrated on two major parts of Kempe's paper; the first expanded on the patch method, applied to Kempe's Figures 1, 15, and 16, and the second dealt with cases where more than three boundaries meet at a point of concurrence.

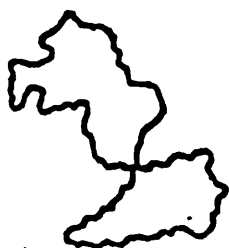


Figure 1



Figure 15



Figure 16

Kempe had denoted the number of districts as D , the number of boundaries as B , and the number of points of concurrence as P at any stage of the development of the patch method, with D' districts, B' boundaries, and P' points of concurrence after the next patch was removed. Story took up the argument that if the next patch had no point of concurrence or line on it when it was removed, an island was then disclosed, and he concluded that, in that case, $P' = P$, $D' = D + 1$, and $B' = B + 1$. However if the patch had no point of concurrence but only a single line, so that when it was removed a peninsula or a district with two boundaries was disclosed, then for the peninsula $P' = P + 1$, $D' = D + 1$, and $B' = B + 2$, and for the district with two boundaries $P' = P + 2$, $D' = D + 1$, and $B' = B + 3$. In the second case Story referred to Kempe's Figure 15, shown above.

Story went on to assert that the preceding 'formulae hold only if the boundaries joined by the line on the patch counted as two before the patch was put on'. He then considered a point of concurrence where boundaries met and when the patch was removed, a district was disclosed with β boundaries. This gave:

$$P' = P + \beta - 1, D' = D + 1, \text{ and } B' = B + \beta.$$

Story concluded that these equations were identical to those of Kempe's (although Kempe used σ rather than β) 'only when three and no more boundaries meet at each point of concourse about the district patched out', giving:

$$P' + D' - B' - 1 = P + D - B - 1.$$

He continued by detailing the alternative situation where the patch had no point of concourse, but only a single line that formed part of the boundary of an island or a district. Removing the patch revealed either Kempe's Figure 16 or Figure 1, shown above. For the island, $P' = P + 2$, $D' = D + 1$, and $B' = B + 2$, and for the district $P' = P + 1$, $D' = D + 1$, and $B' = B + 1$, so in both cases $P' + D' - B' - 1 = P + D - B$.

Story then defined a contour as an aggregate of boundaries, the contour being either simple or complex, according to whether it comprised one, or more than one, district. He asserted that by including contours in the patching procedure, the theorem derived by Kempe could be improved. Kempe's theorem stated that:

In every map drawn on a simply connected surface the number of points of concourse and the number of districts are together one greater than the number of boundaries.

whereas Story's theorem read:

In every map drawn on a simply connected surface the number of points of concourse and the number of districts are together one greater than the number of boundaries and number of complex-contours together.

As Story explained:

If then x of the contours formed by the boundaries of any map are complex, for that map:

$$P + D - B - 1 = x$$

In the second half of Story's paper, Story questioned one of Kempe's claims that:

. . . if we develop a map so patched out, since each patch when taken off, discloses a district with less than six boundaries, not more than five boundaries meet at a point of concourse on the patch.

He claimed that this was valid only when the number of boundaries meeting in each point of concourse does not exceed 3. He then proceeded to detail a procedure to overcome this

restriction. His solution was to use an auxiliary patch whenever more than three boundaries meet, thereby reducing the number of boundaries at a point of concurrence to 3. After carrying out this procedure, the method of patching could be continued as described by Kempe. On completion of the patching, and arriving at a map containing one district and no boundary, Story stated that colouring could commence and the map developed by removing patches, including auxiliary patches, in reverse order. He maintained that by this method 'the map will be coloured with four colours'.

2.5 The consequences

Sylvester believed that during his absence in England there had been an undue delay in the publication of the second volume of the *Journal*, and additionally that previously agreed editorial decisions had been changed and that Story should not have published his note. Sylvester went on to say that it was 'unprofessional', and the relationship between the two scholars became strained. He wrote to the President of Johns Hopkins University, Daniel Gilman, complaining of Story's 'conduct' and of 'disobeying my directions'. In June 1880, Sylvester again, wrote to Gilman asking why Story had not sent him an acknowledgement regarding a paper that Sylvester had sent from England. Then, still aggrieved, Sylvester sent a further letter of eight pages to Gilman on 22 July 1880 [31]. Indeed, such was his annoyance that his haste made parts of the letter illegible. In this letter Sylvester complained that he was not advised of whether the *Journal* had been published, and if so, when. He also objected to his treatment by Story and questioned whether other contributors had received equally poor conduct. Sylvester was so incensed that he formally requested that Story have no further involvement with the *Journal* as he (Sylvester) no longer had confidence in Story. He made it clear that Story could be made aware of his opinion and the contents of the letter.

Gilman mediated between the two, but Story's name did not appear in later issues of the *Journal*. Story resigned from the editorial board and started to look for a new position, a task that took him a number of years. Like all disagreements, it would be wrong to put all of the blame on one party. Sylvester most certainly contributed to the delay in publishing by making late changes to his own paper and rearranging the order of its contents. However, it is worth quoting from a letter dated 7 August 1880 from C. S. Peirce to Gilman [32], which included:

I have received from Sylvester an account of his difficulty with Story. I have written what I could of a mollifying kind, but it really seems to me that Sylvester's complaint is just. I don't think Story appreciates the greatness of Sylvester, and I think he has undertaken to get the Journal into his own control in an unjustifiable degree . . . It is no pleasure to me to intermeddle in any dispute but I feel bound to say that Sylvester has done so much for the University that no one ought to dispute his authority in the management of his department.

There has been some discussion as to who was really the founder of the *American Journal of Mathematics*. Most commentaries give the credit to Sylvester; however, Story proposed the creation of such a publication to Gilman when he was first interviewed for the position at Johns Hopkins. At Sylvester's farewell leaving banquet, on 20 December 1883, Gilman gave the credit to Sylvester. However, Sylvester's response asserted [33]:

You have spoken about our Mathematical Journal. Who is the founder? Mr Gilman is continually telling people that I founded it. That is one of my claims to recognition which I strongly deny. I assert that he is the founder. Almost the first day that I landed in Baltimore . . . he began to plague me to found a Mathematical Journal on this side of the water something similar to the Quarterly Journal of Pure and Applied Mathematics . . . Again and again he returned to the charge, and again and again I threw all the cold water I could on the scheme, and nothing but the most obstinate persistence and perseverance brought his views to prevail. To him and to him alone, therefore, is really due whatever importance attaches to the foundation of the American Journal of Mathematics.

The reality is that Sylvester had the international standing, with links in Europe and previous experience of being involved in the creation of a mathematical periodical, the *Quarterly Journal of Pure and Applied Mathematics* of which he was editor until 1878.

Independently, Story had formulated the idea of a learned mathematical publication and wanted to be involved in its creation. However, without Gilman's continual encouragement, direction, and belief that such a journal would be of great benefit to mathematics in America, it most probably could not have happened as it did in 1878.

Even though Sylvester had left Johns Hopkins at the end of December 1883, Story continued to believe that he needed to move to pastures new, so in 1887 he was relieved to be offered employment at the newly opened Clark University in Worcester, Massachusetts. This additionally allowed him to develop a mathematics faculty according to his own ideas. Although Story was not considered as a great mathematician, the President of Clark believed that he was the best available at that time. Indeed, Story was so successful in his new position that by 1892 it was generally accepted that Clark had the best mathematics department in North America.

Story's situation is best summed up by Roger Cooke and V. Frederick Rickey [22]:

There are many reasons why Story might have wanted to leave Hopkins. He was not a full professor there, though he had been there thirteen years. He was not the editor of the *American Journal of Mathematics*, which had been one of his youthful ideas. Finally, he had come to feel that Hopkins was not the wonderful place intellectually that he thought it might and should be . . . But perhaps most importantly of all, he would have the opportunity to develop a department that focused on graduate education and on research. And he could do it the way that he thought best. For all these reasons, it is likely that the opportunity to move to Clark would have attracted Story.

2.6 Charles Sanders Peirce (1839–1914)

Returning to the history of graph theory in America; C. S. Peirce attended the Johns Hopkins Scientific Association on 5 November 1879 when Story presented Kempe's paper and his own follow-up note.

Charles Sanders Peirce is remembered as a philosopher, mathematician and logician, and for his controversial and unconventional lifestyle. He studied at Harvard University, where his father, Benjamin Peirce, was Perkins Professor of Mathematics and Astronomy.

After graduating, Charles remained there doing graduate research and then worked as an assistant at Harvard Observatory for three years. From 1859, for approximately thirty years, he had a parallel part-time employment with the Coast Survey as an assistant, where his father was the director. Either through choice, or because he was not considered suitable, he did not obtain a permanent position within a mathematical faculty of a university until 1879. This did not prevent him from producing significant seminal work in a wide range of subjects, including probability and statistics, psychophysics (or experimental psychology) and species classification. In addition, he carried out major astronomical research, as well as exploring associative algebras, mathematical logic, topology and set theory.

In the early 1860s, Peirce encountered the four-colour problem, most probably by way of De Morgan's review of Whewell's book in the *Athenaeum*, and it is probable that he was the first American scholar to take an interest in the subject. In the late 1860s, possibly 1869, he presented an attempt to prove that four colours are sufficient to colour a map to the Mathematical Society of Harvard University. His attempt was never published, but he claimed that those who attended the meeting 'discovered no fallacy in it'. His manuscripts, held in the Houghton Library at Harvard University, give no details of his solution to the problem. He later wrote [34]:

About 1860 De Morgan in the *Athenaeum*, called attention to the fact that this theorem had never been demonstrated; and I soon after offered to a mathematical society at Harvard University a proof of this proposition extending it to other surfaces for which the numbers of colours are greater. My proof was never printed, but Benjamin Peirce, J. E. Oliver, and Chauncey Wright, who were present, discovered no fallacy in it.

A second manuscript by Peirce, dated October 1869, includes map colouring in connection with his 'logic of relatives'; this too can be found in the Houghton Library [34]. In the early 1870s, Peirce made a lengthy tour of Europe and in June 1870 visited De

Morgan in London. Although De Morgan was in poor health at the time, it would be logical to assume that their discussions included the four-colour problem [35].

As mentioned earlier, Story was much involved in the four-colour problem, and remained so throughout the rest of his life. Indeed, he was still corresponding on the subject with C. S. Peirce at the end of 1900 and their letters indicate that they, like many others who worked on the four-colour problem, were at times frustrated at their lack of progress. Peirce's approach to the four-colour problem was algebraic and it is more than likely that they discussed the subject during their time together (1879–84), at The Johns Hopkins University. An indication of their continuing efforts and frustration can be seen from their correspondence; a letter from Peirce to Story dated 17 August 1900 [34] included mention that he had found the proof contained in his unpublished paper. In addition, in a double-dated letter from Story to Peirce, the first part dated 1 December 1900 and the second dated 6 December 1900 [34], Story blamed Peirce for the delay in sending the letter. The letter included:

Dec. 1, 1900

... As to my not answering your letter about the four-color problem, I am heartily tired of the subject. I have spent an immense amount of time on it, and all to no purpose. Your first method had occurred to me years ago, but I did not succeed in getting anything out of it.

Dec. 6, 1900

My delay in sending this off is largely your own fault. You have again reminded me of that fascinating but elusive problem, and I have spent the time since writing the above in trying to solve it, but alas! I believe that the case of exception to Kempe's method requires that the map shall have at least one triangle or quadrilateral district, in which case the pentagon is not the next district to be colored, i.e. the exception does not occur. But I cannot prove it ...

It indicates that between the two dates Story had received some communication, possibly a letter, from Peirce suggesting a further approach to the problem that Story attempted without success.

The following extracts from Peirce's writings give a hint to his involvement in the subject and his attempts to provide a proof. These include [34]:

Some time in the early sixties Augustus De Morgan mentioned in the *Athenaeum* that experience showed that four colours would suffice to distinguish confine regions on any map . . . a reproach to logic and to mathematics that no proof had been found of a proposition so simple.

He also believed that because of Cayley's work in logic he must have tried that route to find a solution but had failed. Also included in [34] is the following, which indicates Peirce's involvement in the search for a solution to the four-colour problem:

But his writings over the years are interlaced with references to the problem; his notebooks are full of sketches and diagrams of various regional possibilities reflecting his continuing interest and experimentation. The fragmentary nature of these attempts is evidence of the frustration that never ceased to haunt him.

In 1879, Peirce was appointed a part-time lecturer in logic in the Department of Mathematics at Johns Hopkins University, headed by Sylvester. Initially things went well there and Peirce was exposed to new people and ideas. On 5 November 1879, he attended the meeting of the University's Scientific Association where Kempe's paper was outlined and the minutes record that Peirce contributed to the discussion [36]:

Remarks were made upon this paper by Mr. C. S. Peirce.

Peirce, still working on the four-colour conjecture, presented a paper on the subject at the next meeting on 3 December of that year; no copy of this work has survived but the following record of the meeting included a reference that Peirce [35]:

... discussed a new point in respect to the Geographical Problem of the Four Colors, showing by logical argumentation that a better demonstration of the problem than the one offered by Mr. Kempe is possible.

His years at Johns Hopkins were perhaps his most productive and significant period and he had many papers published in the *American Journal of Mathematics*, with at least one (in 1878) at the request of Sylvester. The following appears in an article by Carolyn Eisele [37]:

A year earlier Peirce had been invited by Sylvester to publish a paper on map projections in the forthcoming issue of the *American Journal of Mathematics*.

Peirce's employment at the Johns Hopkins University was not to be a long tenure. In 1884 shortly after Simon Newcomb was appointed professor of mathematics and astronomy there, he felt that it was his duty to inform the University's trustees that Peirce had been living with his mistress whilst still being married to his first wife. The consequence was that Peirce's contract was not renewed, so that the University would not attract scandal by association. Peirce was never to hold another academic post. In October 1876, Peirce had separated from his wife of thirteen years and embarked on a path that would cause him considerable discomfort and greatly affect his career. A short while later he set up house with a French gypsy, Juliette Froissy Pourtalès. He was divorced from his first wife on 24 April 1883, and married Juliette six days later. Newcomb was a contemporary of Peirce, and had also studied at Harvard under Benjamin Peirce and graduated the year before Charles Peirce. Over the years, Newcomb and Charles Peirce kept up an active correspondence which displayed a mutual respect, if not a closeness, between the two. Nevertheless, this relationship did not prevent this unfortunate event.

In later years, Peirce became increasingly more withdrawn from public life and colleagues and more erratic in his behaviour. He did however present a paper *The map-coloring problem* at the Scientific Session of a meeting of the National Academy of Sciences, held in New York City on 14-15 November 1899. Again, no manuscript of this work has survived [38].

Conclusion

Through the efforts of scholars such as Sylvester, Story and Peirce, a climate was provided for American-born and American-educated mathematicians to make significant contributions to mathematics, and to trees and the four-colour problem in particular.

Additionally the support of graduate schools, Johns Hopkins at first, then Harvard, Yale, Princeton, and others, advanced the development of American mathematics and postgraduate study that continued through the 1890s and into the twentieth century. By 1910, America had several native professors who had received all their training in the USA and would go on to earn international reputations. This healthy state of affairs in American mathematics would pave the way for the USA to become the leading mathematical country by the middle of the twentieth century.

Additionally, Kempe's paper provided the impetus for other mathematicians, mostly American, to develop his ideas, two of which were later to be called an 'unavoidable set' and a 'reducible configuration'.

The solid foundations laid, and the encouragement given by the leading mathematicians around the turn of the century, provided a foundation for American mathematicians to make their mark. Near the beginning of the twentieth century, American mathematicians were beginning to take a serious interest in graph theory and the four-colour problem. This move was initiated and influenced by Oswald Veblen and George David Birkhoff (see Chapter 4).

Before evaluating the major contributions of some important American mathematicians, the next chapter reviews the work done in Europe in graph theory from 1890 to 1930.

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Chapter 3

Graph theory in Europe (1880–1930)

While scholars in the United States of America were beginning to take an active interest in graph theory in the last quarter of the nineteenth century, work on the subject continued in Europe. Peter Guthrie Tait endeavoured to improve on Kempe's work on the four-colour map problem and also studied the colouring of the edges of countries. Percy John Heawood published a major paper in 1890, in which he located the mistake in Kempe's paper of 1879 (see Chapter 2), and discussed the colouring of maps on surfaces. Later contributions were by Lothar Heffter (1891), who worked on the colouring of maps on orientable surfaces, and Heinrich Tietze (1910), who studied the colouring of maps on certain non-orientable surfaces. Julius Petersen, Paul Wernicke, Hermann Minkowski, Kazimierz Kuratowski and Alfred Errera were among other Europeans who were interested in graph theory and the four-colour problem.

Further information of several of the topics and mathematicians covered in this chapter can be found in [1].

3.1 Peter Guthrie Tait (1831–1901)

Tait, a Scottish applied mathematician, who was acquainted with Hamilton and Kirkman (see Appendix I), was an accomplished teacher and productive author. Having learned of the four-colour problem from Cayley [1], Kempe's supposed improved solution to the problem stimulated his interest [1]. Tait advanced a number of 'more simple solutions than Kempe's', in two papers one presented to the Royal Society of Edinburgh in 15 March 1880 [2] and the second [3] in July of that year. These more simple solutions were based on the Kempe's incorrect proof. The first paper included suggestions of approaches that could be used to provide alternative proofs. This paper turned out to be of little merit.

Undeterred, he was back before the society in July with the second paper which expanded upon an idea that was fleetingly mentioned in his previous one. Tait observed that utilising four colours to colour the countries of a cubic map is equivalent to using three colours to colour the edges of the countries so that all three colours meet at each vertex. Although interesting, this substitute problem was found to be as difficult to prove as the original four-colour conjecture. However, this was the first suggestion of an alternative form of the problem. This colouring became known as a *Tait colouring* or an *edge colouring*; giving rise to the conjecture ‘The edges of a bridgeless cubic planar map are three-colourable’. Many later equivalent forms of the four-colour conjecture were based on it.

In a paper [3] published in 1880, Tait returned to the idea of colouring boundary lines, this time focusing on cubic maps. His ‘theorem’ was that every cubic polyhedron has a Hamiltonian cycle. Of this Kirkman commented [4]

A rigorous demonstration is much to be desired, flowing from the simple definition of a p-edron with triedral summits. But I share the opinion of Professor P. G. Tait, that our prospect of obtaining such demonstration is very remote indeed. It is my impression that he knows more about these circles than any other.

Tait reiterated this assertion in a paper published four years later [5] but was unable to prove that every cubic map had a Hamiltonian cycle, and this was later shown to be false. It was shown to be false by William Thomas Tutte (see Chapter 9) using a counter-example with forty-six vertices, but he observed that if the graph of a cubic map has a Hamiltonian cycle the map could be coloured with four colours.

3.2 Percy John Heawood (1861–1955)

In 1890 a major error in Kempe’s proof of the four-colour theorem was announced by Percy Heawood from Durham [6], although his words were somewhat apologetic. Although he is remembered mainly for uncovering the flaw in Kempe’s arguments, he was a fine scholar and serious contributor to map colouring for nearly sixty years. In his paper, he

modified Kempe's chain method to show that five colours are sufficient for colouring any map on the plane or sphere.

In addition, in this paper Heawood took up the question that had been suggested by Kempe of finding the number of colours required to satisfy the normal criteria of colouring maps on the sphere and other closed surfaces, including the torus; and presented an upper bound for the chromatic number of a map embedded on a given surface. The paper also included a proof that:

... seven colours are necessary and sufficient to colour all maps on a torus.

Orientable surfaces (2-sided surfaces) can be classified by their *genus*. An orientable surface is of genus g if it is topologically homeomorphic to a sphere with g handles. Examples of orientable surfaces are the sphere ($g = 0$) and the torus ($g = 1$). There are also non-orientable surfaces (1-sided surfaces), also classified by their genus. Examples of non-orientable surfaces are the projective plane ($q = 1$) and the Klein bottle ($q = 2$).

The *chromatic number* of a surface S is the smallest number of colours that are required to colour all maps on S so that neighbouring countries are differently coloured. In his paper Heawood tried to establish the chromatic number of any orientable surface S_g , a torus with g holes, which gives S_g for all positive numbers g . He asserted that for, $g \geq 1$, every map on S_g can be coloured with $H(g)$ colours, where:

$$H(g) = \lfloor \frac{1}{2} (7 + \sqrt{1 + 48g}) \rfloor$$

The square brackets are to indicate that the result should be rounded down if it is not an integer. For example for a two-holed torus, the number of colours is:

$$H(2) = \lfloor \frac{1}{2} (7 + \sqrt{97}) \rfloor = \lfloor 8.4244... \rfloor = 8.$$

Regrettably he was unable to prove that for $g \geq 2$ there are maps on S_g that require the number of colours calculated by his formula. He proved that $H(g)$ colours are sufficient, but not that this many were required. His claim later became known as the Heawood Conjecture;

For each positive number g , there is a map on the surface of an g -holed torus that requires $H(g) = \lceil \frac{1}{2}(7 + \sqrt{1 + 48g}) \rceil$ colours.

It would be a further 78 years before it was proved by Ringel and Youngs (see Chapter 9).

It was also in his 1890 paper that Heawood addressed the Empire Problem; how to colour a map which includes several empires each comprising a *mother country* and any number of *colonies* — the mother country and its dependent colonies all being coloured the same. He presented a number of examples and posed the question if ‘any country may exist of two distinct portions but not more’ how many colours are needed. Using Euler’s formula he demonstrated that the number of colours required does not exceed 12 and more generally for a country comprising r distinct portions, r greater than 1, the number of colours required does not exceed $6r$.

Heawood’s revelation of Kempe’s error was not widely known for several years, and Heawood received little recognition for his paper. Not only did many people fail to accept his correction at the time, but six years later, when Charles-Jean-Gustave-Nicolas de la Vallée Poussin (1866–1962) rediscovered the fault in Kempe’s solution, no mention was made of Heawood’s priority. It is also on record that certain later mathematicians assumed that Kempe had solved the four-colour problem and even based their own work on his ‘proof’ [7].

Heawood’s second paper (1898) on map colouring [8], included the observation that if the number of edges around each region of a regular map is divisible by 3, then the regions can be coloured with four colours. He elaborated this idea as a general result. To each vertex of a cubic map (a map where exactly three edges meet at each vertex) he assigned the values $+1$ and -1 such that the sum of the values of the three edges of each region is divisible by 3; if this is achieved then the map can be coloured with four colours. As Tait had found, a colouring of a cubic map with four colours is equivalent to colouring its edges

with three colours; if the vertices of a cubic map are labelled v_1, v_2, \dots, v_n , then a system of congruences of the form

$$x_i + x_j + \dots x_k \equiv 0 \pmod{3}$$

can be generated with one congruence for each region. Each of the unknowns x_i is either $+1$ or -1 , and x_i appears in the congruence corresponding to a region if and only if v_i is on the boundary of that region.

This paper, together with later ones in 1932 [9] and 1936 [10], explored systems of congruences associated with map colouring. The 1898 paper included a proof that the regions of a cubic map can be coloured with three colours if and only if each face has an even number of boundary edges, which had already been mentioned in Kempe's paper.

Almost fifty years after his 1890 paper, Heawood was still pursuing map colouring. In his 1936 paper, he developed an estimate for the probability that the four-colour conjecture was true, and although his argument was rather loose, he gave an indication that the probability of failure did not exceed $e^{-4n/3}$, where n is the number of countries. At the time of publication it had been proved that all regular maps with twenty-seven countries could be coloured with four colours, and so his estimate would have been $e^{-36} \approx 0.67 \times 10^{-12}$, a very small probability.

In two notes, in 1943 [11] and 1944 [12], Heawood published corrections to his 1936 paper. In his 1936 paper Heawood had conjectured that, if two adjacent regions of a complete map were left out of account, the congruences (mod 3) corresponding to the remaining regions could always be solved, no matter what constraints were assigned to these remaining regions. However, in the first part of his 1943 paper he presented an example that showed that this conjecture is not true. If it had been true then the four-colour theorem would necessarily have been proved. Daniel Clark Lewis Jr. (see Chapter 8) reviewed the 1943 [13] and 1944 [14] papers and in the later review Lewis wrote:

The subject of map congruences led the author to the following problem. Consider a sequence of n (not necessarily distinct) symbols, each one of which is either a 1, 2 or 3. Let U_n denote the set of all

3^n such sequences. Let V_n denote any subset of U_n such that every sequence of U_n “avoids” at least one sequence in V_n . When we say that one sequence “avoids” another, we mean that every symbol in one sequence is different from the symbol in the corresponding place of the other sequence. Let $f(n)$ denote the minimum number of sequences which such V_n must have. The problem is to evaluate $f(n)$. It is easily shown that $f(1) = 2, f(2) = 2, f(3) = 5, f(4) = 9, f(5) = 16$, and that $f(n) < 2f(n-1)$. The author gives an upper bound for $f(n)$... It is not known whether the author’s upper bound is actually equal to $f(n)$.

At the age of nearly 90, Heawood published his final paper on map colouring [15]. On this occasion it was reviewed by a mathematician important to this narrative, Philip Franklin (see Chapter 5) [16]. The paper explored an arithmetic problem equivalent to the four-colour conjecture, given certain restrictions that make the problem correspond to a map. Heawood developed the arithmetic and attempted to show that, for a number of cases, it can be solved without restriction. He did not propose any proof, but suggested that the work could lead to a method for solving the four-colour problem.

3.3 Lothar Wilhelm Julius Heffter (1862–1962)

The German mathematician Lothar Heffter studied the colouring of maps on orientable surfaces. In his 1891 paper [17], he constructed maps in which every face meets every face, and he called this a *system of neighbouring regions*. The problem being considered was that of embedding the complete graph thus indicating that the genus of the surface is the Heawood number. It was in this paper that the first real use of dual graphs was made, although Kempe had mentioned the concept earlier.

The deficiency in Heawood’s argument was pointed out by Heffter, who showed that the Heawood conjecture is true for $g = 2, 3, 4, 5, 6$, and a few other values, although he was not able to provide a general proof. His proof involved fixing the number of neighbouring regions n and varying the genus p , and then investigating the least value of p

such that n neighbouring regions could be constructed on the two-sided surface S_p . His paper stated:

We next enquire after the minimal value of the genus of a surface which does admit n neighbouring points, and call this number p_n ; that is to say it will be supposed that on a surface of genus p_n , there really exist n neighbouring points, but not on one of lower genus. Then it is easy to determine a lower bound for p_n .

His paper continued by stating that the n neighbouring points and their $\frac{1}{2}n(n-1)$ connecting edges give a system of n vertices and a number of faces (all simply-connected) on a surface. Heffter developed an upper bound for the number F of simply-connected faces, arguing that $n-1$ faces surround each of the n points and that the faces are at least triangles. In his paper he used the generalised form of Euler's polyhedral theorem for S_g ,

$$V - E + F = 2 - 2g,$$

where V is the number of vertices, E the number of edges and F the number of faces and stated:

For the number F of these simply-connected faces, we obtain the upper bound

$$F \leq \frac{n(n-1)}{3};$$

this is because around each of the n points lie $n-1$ faces, which certainly are at least triangles, since if digons were to be found two of the n points would be doubly joined with each other. Now we apply to the system of vertices, edges and faces of the surface of genus p_n the generalized Euler polyhedral theorem, thus obtaining

$$2p_n - 2 = \frac{n(n-1)}{2} - n - F,$$

$$2p_n - 2 \geq \frac{n(n-1)}{2} - n - \frac{n(n-1)}{3},$$

$$p_n \geq \frac{(n-3)(n-4)}{12},$$

and since in every case p_n must be an integer, we can write

$$p_n \geq \frac{(n-3)(n-4) + 2\alpha_n}{12},$$

where $2\alpha_n$ is the smallest positive integer which makes the numerator divisible by 12.

Although Heffter was unable to provide a complete general proof of $\chi(S_p) = H(p)$ for infinitely many values of p , it is significant that the above expression includes the number 12 as the eventual proof of the Heawood conjecture in 1968 (see Chapter 9) [18] was split into twelve separate cases, depending on the remainder when n is divided by 12.

3.4 Heinrich Franz Friedrich Tietze (1880–1964)

Another mathematician working in a similar vein to Heffter was the Austrian Heinrich Tietze who was later highly regarded for his book *Famous Problems of Mathematics* [19]; this book included colour plates illustrating the colouring of maps on different surfaces.

His 1910 paper [20] developed Heffter's approach, but for non-orientable surfaces, employing Heffter's terminology of neighbouring regions and applying the arguments of Heawood; he found an upper bound for $\chi(N_q)$, the chromatic number of the surface N_q , for $q \geq 2$, the upper bound is:

$$\chi(N_q) \leq \lceil \frac{1}{2} (7 + \sqrt{1 + 24q}) \rceil,$$

Applying Euler's generalised polyhedral formula in this case gives:

$$q - 2 = \frac{n(n-1)}{2} - n - F,$$

$$q - 2 \geq \frac{n(n-1)}{2} - n - \frac{n(n-1)}{3},$$

so

$$q \geq \frac{(n-3)(n-4)}{6} \quad (n \geq 7).$$

The Möbius strip is a distinct case in that it is one-sided, but it has a boundary. In his paper, Tietze showed 'that it is possible for six, but no more, neighbouring regions to be marked on a Möbius strip'. He also concluded that the chromatic number $\chi(N_1)$ of the projective plane N_1 is 6, but was unable to determine the chromatic number of the Klein bottle N_2 .

3.5 Julius Peter Christian Petersen (1839–1910)

Julius Petersen was a Danish mathematician who published papers on graph theory in the 1890s and later. Many of his ideas would spark the interest of a number of young American mathematicians. He had regular correspondence with Sylvester [21], on map colouring and on the factorisation of graphs.

Petersen's work was principally in geometry, but his pioneering paper on the factorisation of regular graphs was perhaps his most important publication. Published in 1891, the paper [22] was the first to describe a theoretical approach to the problem of factorising regular graphs. He showed that all such graphs of even degree could be resolved into 2-factors (graphs regular of degree 2).

In his paper, Petersen commented that the factorisation of regular graphs, when applied to graphs of odd degree, was more challenging than when applied to those of even degree. However he continued by exploring graphs of odd degree and proved a significant result regarding cubic graphs:

Provided that a graph has no more than two leaves, then a regular graph of third degree always possesses a 1-factor.

(A leaf is a part of a graph which can be disconnected from the remainder of the graph by removing a single edge).

He also recalled that Tait had demonstrated that the four-colour conjecture is equivalent to a conjecture about the colouring of edges. He developed Tait's methods and showed that the four-colour conjecture is related to Hamiltonian cycles and the traversality of certain graphs, giving rise to the following conjecture:

In a bridgeless cubic map, it is possible either to tour all the vertices by a Hamiltonian cycle, or to find a group of disjoint even-length cycles covering all the vertices.

Petersen also published two short notes on cubic graphs — one in 1898 [23] and one in 1899 [24]. In the first he proved the theorem:

An indecomposable graph of the third degree must have at least three leaves.

He restated Tait's theorem as:

A graph of the third degree which has no leaves can be decomposed into three graphs of the first degree.

He went on to say that:

... a graph of the third degree which has no leaves can be decomposed, either into three graphs of the first degree, or into one of degree one and one of degree two.

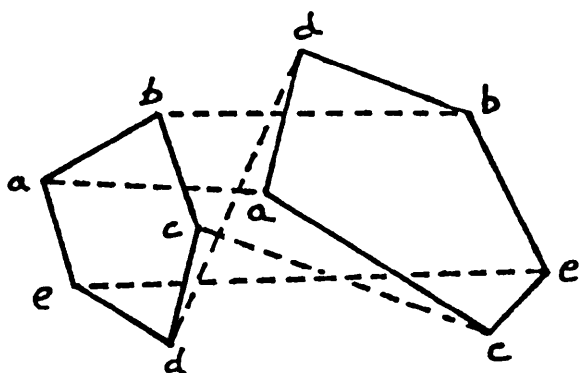
In my terminology, Tait's conjecture can be stated as follows:

A graph of the third degree which has no leaves can be decomposed into three graphs of the first degree.

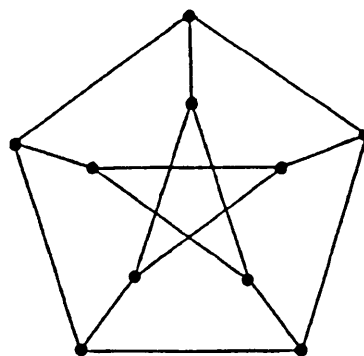
and commented that this is:

... stronger than my theorem and it seems to me impossible that it can be true without the adjunction of extra conditions. What is more, as I shall show, I have succeeded in constructing a graph to which Tait's theorem does not apply.

This graph now called the *Petersen graph*, shown below, is the most notable part of his 1898 note (although Kempe had published a version of the graph twelve years earlier).



*The Petersen graph
(taken from his paper)*



*An alternative illustration of the
Petersen graph showing its symmetry*

In the second paper he commented that 'M Kempe only skimmed over the problem; he committed his error just where the difficulties began' and went on to say somewhat unexpectedly, as he failed to give any reason:

I know nothing with certainty, but if it came to a wager I would maintain that the theorem of the four colours is not correct.

3.6 Paul August Ludwig Wernicke (1866 or 1868–c1940)

Another German whose imagination was captured by the four-colour problem was Paul Wernicke; who was born in Berlin. He was Professor of Modern Languages at the State College of Kentucky between 1894 and 1906; however, during this time he turned to mathematics and studied under Hermann Minkowski (see 3.7) obtaining his doctorate in that subject from Göttingen University in 1903.

It is interesting to note that he was elected a member of the American Mathematical Society in 1897 as Professor Wernicke, whilst he was at the State College of Kentucky, but the entry in the Society's records does not state the discipline of his subject. His first notable contribution to mathematics was as a delegate to the Fourth Summer Meeting of the American Mathematical Society held in Toronto in 1897, which was chaired by the Society's president Simon Newcomb. Wernicke gave a lecture or presented a paper entitled *On the solution of the map-color problem*, which has not survived. The Society included an abstract of the work in their *Bulletin* [25], but it does not give a clear description of the paper's contents. From the abstract, it appears that the paper or lecture was concerned with Tait's relationship connecting the 3-colouring of the edges of a cubic map with that of the 4-colouring of the countries. Wernicke's method appeared to be that of adding new countries to a map to convert it into one that he could colour with four colours. The abstract in the *Bulletin* included:

Given a map correctly colored and with its frontiers marked, the author proves that any triangles, quadrangles, and pentagons can be introduced and correctly marked at the same time. The main theorem then follows by induction.

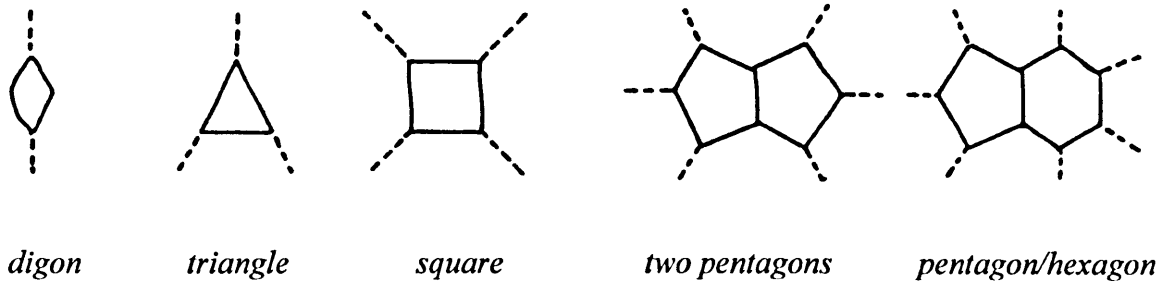
It is considered that this approach had little merit [7].

He must have returned to the USA after obtaining his doctorate and here he published his main paper [26] in 1904 which was written in Göttingen in May 1903. In it he proved that a cubic map that contained no digon, triangle, or quadrilateral must contain, not only a

pentagon (as shown by Kempe), but also either two adjacent pentagons or a pentagon adjacent to a hexagon. This can be expressed, for all F , as:

In a cubic map which satisfies $\rho^*(F) \geq 5$, where $\rho^*(F)$ represents the number of boundary edges in the faces F , then there must be a pentagon touching another pentagon or a hexagon.

This extended the list of country shapes that form unavoidable sets to:



This paper presaged the interest in unavoidable sets in maps and was the most significant advance in the field since the work of Kempe and Heawood.

3.7 Hermann Minkowski (1864–1909)

Minkowski was another mathematician caught up in the quest for a solution to the four-colour problem. He was a Lithuanian-born naturalised German who taught for a while at Göttingen University, originated the geometry of numbers, and worked on mathematical physics and the theory of relativity.

For all his brilliance, he was unable to solve the four-colour problem. It is recorded [27] that he interrupted a topology lecture he was giving to inform his student audience that the four-colour conjecture had not been solved because only ‘third-rank’ scholars had addressed it. He further informed the students that he believed that he could prove it. The story is described in [7]:

Finally one rainy morning, Minkowski entered the lecture hall, followed by a crash of thunder. At the rostrum, he turned towards the class, a deeply serious expression on his face. ‘Heaven is angered by my arrogance’, he announced. ‘My proof of the Four-colour Theorem is also defective’. He then took up the lecture on topology at the point where he had dropped it several weeks before.

3.8 Kazimierz Kuratowski (1896–1980)

Kuratowski was a Polish mathematician whose work was mainly in the area of point-set topology but who made a significant contribution to graph theory in his paper published in 1930 [28]. The paper contained the following theorem which is now called Kuratowski's theorem:

If G is a non-planar graph, then it contains a subgraph which is a subdivision of K_5 or $K_{3,3}$.

The American Orrin Frink (1901–1988) and a colleague, P A Smith, independently arrived at the same conclusions as Kuratowski, but did not publish their results as the details were too close to those of Kuratowski. Frink published work in many areas of mathematics, including lattice theory and topology, and is remembered by graph-theorists mostly for his simplified proof of Petersen's theory, published in 1926 [29], which was described by Dénes König [30]:

Finally Frink succeeded in reducing the theorem in an elegant and comparatively easy way ... in which, in contrast to all his predecessors, he was able to avoid any counting process.

3.9 Alfred Errera (1886–1960)

Alfred Errera was a Belgian mathematician who spent most of his career at the University of Brussels and who made significant contributions to graph theory and map colouring, in parallel with his American colleagues with whom he was in correspondence. He began an interest in map colouring in the 1920s and corresponded with Heawood on the subject during that decade. In any event, Errera was a lone Belgian voice in graph theory, but published many papers on the subject, several of which contained significant contributions. During the 1920s he was working along similar lines as American mathematicians and his graph-theoretical papers contained many references to them.

His 1921 paper [31] dealt with the use of chains for attacking the four-colour problem. His 1922 paper [32] provided another simple proof of Petersen's theorem. In 1923 he published a paper [33] on planarity where he addressed the utilities question:

In the plane are three houses and three wells: the problem is to join every house with every well by nine paths in all, so that no two of these paths cross each other.

He proved the generalisation:

If the points $A_1, A_2, \dots, A_u; B_1, B_2, \dots, B_v$ lie in a plane, then exactly $2u + 2v - 4$ of the uv edges $A_i B_j$, and not more, can be drawn without any two of the edges crossing each other in the plane ($u > 1, v > 1$).

Over the next five years, he returned to map-colouring problems with one paper in 1923 [34], two in 1924 [35] [36], one in 1925 [37], and one in 1927 [38]. The highlights of these papers included the following.

In his second paper of 1924 [36], presented at the International Congress of Mathematics held in Toronto, he started with a brief overview of the four-colour problem mentioning Cayley, Kempe, Heawood, Petersen and Tait. He outlined the results of Birkhoff's 1913 paper *The reducibility of maps* and Franklin's 1922 paper *The four color problem*. Errera stated that a fuller version of his summary could be found in his 1925 paper [37]. In this latter paper he proved that an irreducible configuration must contain at least thirteen pentagons. The concept of reducibility is explained clearly in [1] as:

If there are plane maps which need five colours, then there must be among them a map with the smallest number of regions; such a map is said to be **irreducible**. The basic idea is to obtain more and more restrictive conditions which an irreducible map must satisfy, in the hope that eventually we shall have enough conditions either to construct the map explicitly, or, alternatively, to prove that it cannot exist.

This paper also included a proof that maps with only vertices of degree 5 and 6 cannot occur in a minimal counter-example to the four-colour theorem. His 1927 paper [38] like his second paper of 1924 [36] gave a comprehensive history on the four-color problem.

After World War II Errera revisited colouring problems and published papers in 1947 and 1948. In the first [39], he explored the decomposition of a regular map into concentric

rings indicating how this may relate to the four-colour problem. In the second [40] he developed formulae for the number of ways of colouring particular configurations of graphs. In 1950, he published two further papers [41] [42] both of which were reviewed by Tutte (see Chapter 9). In these Errera investigated the classification of cubic maps on the sphere and discussed the reducibility of cubic maps being coloured with four colours using Tait's edge-colouring method. In the second paper, he formulated a new version of the four-colour conjecture by fusing Heawood's expression in terms of congruences (mod 3) with Whitney's theorem on Hamiltonian cycles (see Chapter 6).

In a paper of 1952 [43], Errera addressed the classification of polyhedra, again building on work by Whitney. Also in the same year he published an expository paper [44] reviewing the problems associated with the four-colour problem. In another paper [45], but published a year later, he returned to the classification of spherical polyhedra and again cited the work of Whitney.

Conclusion

So although many European mathematicians had devoted significant time and effort to the four-colour problem for over seventy years, no solution had been found. However, in that time considerable work had been published on graph theory. Europeans had carried out nearly all of this work, but this situation was soon to change as American mathematicians began to make their mark on the development of the subject.

The next chapter covers the graph-theoretic work of the first Americans to make major contributions to the subject — Oswald Veblen and George David Birkhoff.

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Chapter 4

Oswald Veblen and George Birkhoff

The first American mathematicians to publish important work on graph theory were two of the most notable men in American mathematics in the early twentieth century. They were Oswald Veblen and George David Birkhoff, and between their first meeting in 1902 and the latter's death in 1944 they remained friends and colleagues in advancing mathematics in the USA.

In the volume of the American Mathematical Society's publication in which Veblen's first paper on graph theory appeared, so too did the first significant one on the same subject by Birkhoff. These two papers, and a follow-up paper of Birkhoff, were to provide the impetus for the ever-increasing contributions from American mathematicians to graph theory and map colouring.

4.1 Oswald Veblen (1880–1960)

Oswald Veblen attended the University of Iowa where his father was professor of mathematics and physics, graduating with an AB degree in 1898, and staying there for a further year undertaking work in the physics department as an assistant. He then went to Harvard University for a year, earning a second AB in 1900. On leaving Harvard, Veblen spent three years at the University of Chicago, gaining much of his early mathematical training and knowledge from an inspiring trio of scholars – his supervisor Eliakim H Moore, Oskar Bolza (1857–1942), and Heinrich Maschke (1853–1908). Veblen's doctoral dissertation, entitled *A System of Axioms for Geometry*, was inspired by Poincaré and written under the direction of Moore, and earned him his PhD degree in 1903. Veblen taught mathematics at Princeton University from 1905 to 1932, initially as a preceptor, and

then as a full professor from 1910. During the academic year 1928–29, he taught at Oxford University as part of an exchange arrangement with G H Hardy.

Veblen's most significant contributions to graph theory were his 1912 paper *An application of modular equations in analysis situs* [1] and the book published in 1922, *The Cambridge Colloquium 1916, Analysis Situs* [2]. Referring to Veblen's book, the graph-theorist W T Tutte (see Chapter 9) was later to remark [3]:

I learned a little Combinatorial Topology at Cambridge. That subject dealt with structures called " n -complexes". These were made by fitting together units called " n -simplexes", n -dimensional analogues of the point, the segment, the triangle and the tetrahedron. The graphs of Graph Theory appeared as the case $n = 1$, and by combinatorial topologists they were usually looked upon as trivialities. However O. Veblen devoted part of a textbook to them and to their associated maps on surfaces.

What made Veblen disposed to apply himself to graph theory can only be speculation. It is possible that during his time at the Universities of Iowa and Chicago he attended meetings of the American Mathematical Society, where papers on the subject were presented. Additionally, he would have had access to learned journals and may have read articles such as Wernicke's 1904 paper. He may also have heard C S Peirce's unpublished presentation *The map-coloring problem* in 1899. Certainly, reports of these meetings would have been available to him. There does not seem to be evidence of Veblen having any contact or correspondence with Peirce or Wernicke, who were known to be working on map colouring around the turn of the century.

The First World War interrupted Veblen's career, and when he returned to Princeton he quickly became regarded as a leading geometer. Because of his work, many graduate students applied to study there or to be employed by the mathematics faculty; one of these students was Philip Franklin (see Chapter 5); the two had worked together during their war service. Veblen's research and influence ranged over many areas of mathematics, including the foundations of geometry and topology, relativity theory and symbolic logic.

Through the work of Veblen and his students, Princeton became one of the leading centres of topology; as a result, he earned the rare designation ‘statesman of mathematics’ around the world, a description found in many articles on Veblen [4]. Although Veblen only published two papers on graph theory they were influential — the best introduction to the subject for many years.

Every few years two prominent mathematicians were selected by the American Mathematical Society to give a series of summer lectures to the membership one of the highest recognitions of a mathematician in America. These meetings were well attended, giving rise to considerable discussion. As was usual with the Colloquium Lectures the papers were revised, extended, and then published as a monograph. Because of the First World War and the Society’s lack of funds, the publication of Veblen’s lectures was delayed. They eventually appeared in 1922 and gave the first comprehensive description of the fundamental concepts of topology and greatly assisted the advance of modern combinatorial topology.

Following his successful AMS lectures, Veblen turned his attention to work other than graph theory. Subsequent to the publication of Einstein’s *General Theory of Relativity* in 1915, he became interested in differential geometry, and from 1922 most of his publications were on this subject and its connections with relativity. This work led to important applications in relativity theory, and atomic physics made some use of his work.

Veblen was much involved in the recruitment into the American academic world of many notable foreign mathematicians. These included Richard Courant (1888–1972), founder and director of the Mathematics Institute at Göttingen University, who was dismissed by the Nazis in 1933 and went to the USA via England in 1934. He became director of the Institute of Mathematical Sciences at New York University. In 1938, the historian of mathematics Otto Neugebauer (1899–1990), joint director of the Mathematics Institute at Göttingen University, was offered a chair at Brown University, arranged by

Veblen, becoming Professor of the History of Mathematics there; after his retirement in 1969 he worked at the Institute for Advanced Study at Princeton where he was made a permanent member in 1980. In addition, Hermann Weyl (1885–1955), a professor of mathematics at Göttingen University moved to the Institute for Advanced Study in 1933, where he remained until his retirement in 1952. Veblen’s work in these matters earned him considerable respect and not a little gratitude.

Just before Veblen gained his doctorate at Chicago, George David Birkhoff enrolled there as an undergraduate. His son Garrett Birkhoff (1911–1996) later recalled [5] that his father:

... entered the University of Chicago in 1902. There he soon began a lifelong friendship with Oswald Veblen, a graduate student who had received an AB from Harvard (his second) two years earlier.

It may seem surprising that, despite their very different personalities and their disagreements on at least two major issues (the encouragement of European immigration of scholars during the 1930s and the initiation of *Mathematical Reviews* in 1940), they remained close friends. It is likely that Veblen was influential in Birkhoff’s early work and selection of research subject matter. It is certainly true that when Veblen’s paper, *An application of modular equations in analysis situs*, was published in 1912, algebraic topology (or *analysis situs* as it was called then) was not widely pursued in the USA.

After his death in 1960, the American Mathematical Society founded an *Oswald Veblen Prize in Geometry*, although the first seven years’ recipients were all rewarded for their work in topology. It was not until 1976 that any work in geometry was found to be worthy of the prize [6].

4.2 *An application of modular equations in analysis situs* [1]

Veblen’s paper on modular equations in analysis situs was presented to the American Mathematical Society on 27 April 1912. It was the first substantial contribution to the four-

colour problem by an American mathematician, and the ever-increasing development of the subject by American scholars was essentially due to him and Birkhoff. In his paper, he drew on ideas from finite geometry and incidence matrices over a finite field. He represented them by matrices by arbitrarily numbering the vertices, edges and countries of a map.

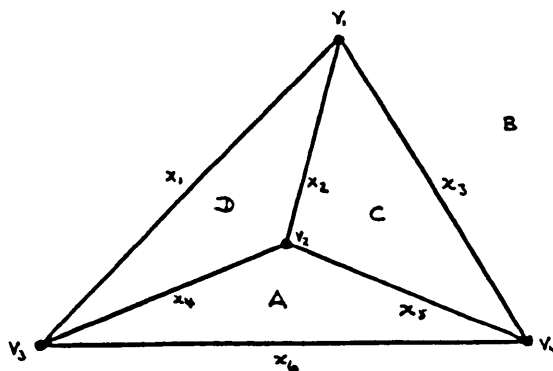
His paper attempted to clarify, using matrices, those linear equations in a finite field that could result in a solution of the four-colour problem. He intended that these equations display the basic properties of a map; he said that they would provide ‘an ... easy proof ... of Euler’s formula’ and in his argument he used the rank + nullity theorem. His fundamental premise was that ‘a map could be ... described by means of two matrices’. He also gave credit to Poincaré, noting that:

These matrices are identical on interchanging rows and columns with those employed by Poincaré, if the + and – signs used by the latter are omitted. [7]

In the following example, matrix A represents the incidences of vertices, numbered v_1 to v_4 , as rows, and edges, numbered x_1 to x_6 , as columns; matrix B represents the incidences of edges as rows, and countries, numbered A, B, C, D , as columns. In matrix A , 1 appears where a vertex is on an edge, otherwise 0 appears; and for matrix B , 1 is used where an edge is a boundary of a country; otherwise 0 is entered. As his example Veblen used:

... the map obtained by projecting an inscribed tetrahedron from one of its interior points to the surface of a sphere ...

The following represents this:



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

the elements of these matrices are to be regarded as integers (mod 2).

Veblen observed that four sets of linear homogeneous equations could be developed from these matrices. He stated that:

In each case the variables and coefficients are regarded as integers reduced modulo two. In other words, let us add according to the rules $1 + 1 = 0$, $1 + 0 = 1$, $0 + 1 = 1$, $0 + 0 = 0$; and multiply according to the rules $1 \times 1 = 1$, $1 \times 0 = 0$, $0 \times 1 = 0$, $0 \times 0 = 0$. All the formal laws of elementary algebra are satisfied by this field.

The first set (1) corresponds to the rows of the matrix A , Veblen explained that:

There is one variable for each edge of the map and one equation,

$$(1) \quad x_a + x_b + x_c + \dots = 0$$

for each vertex, the variables in the equation representing the edges which meet at a corresponding vertex. A solution of this system of equations represents a way of labelling the edges of the map with 0's and 1's so that there shall be an even number of 1's on the edges at each vertex. *The edges labelled with 1's in this manner form a number of closed circuits no two of which have an edge in common.* For let us start with an arbitrary edge labelled 1 and describe a path among the edges labelled 1. Whenever there is by which this path approaches a vertex, since the number of 1-edges at this vertex is even, there is a 1-edge by which the path can go away. Hence the path may be continued till it intersects itself. A portion of the path then forms a closed circuit. It this be removed there are still an even number of 1-edges at each vertex. Another circuit may be removed and so on till all the 1-edges are accounted for.

If (x_1, x_2, \dots, x_a) and $(x_1', x_2', \dots, x_a')$ are solutions of the equations (1) it is clear that $(x_1 + x_1', x_2 + x_2', \dots, x_a + x_a')$ is also a solution. The boundary of each of the α_2 countries of the map is represented by a solution in which each edge of the boundary is marked with a 1 and each other edge with a 0. One such solution is supplied by each column of the matrix B .

Veblen called these ‘fundamental solutions’. He pointed out that the number of linearly independent solutions of the first set of equations is the number of countries less one (that is, $\alpha_2 - 1$), and that the total number of solutions is $2^{(\alpha_2 - 1)}$.

The second set of equations (2) is developed from the columns of the matrix A , being $v_a - v_b = 0$, for each edge $v_a v_b$. Veblen goes on to say:

The only possible solutions are such that all the variables are equal. For if v_a is given, v_b must be equal to v_a ; if v_c is connected with v_b by an edge, v_c is also equal to v_a , and so on. Since there is a path along the edges joining any vertex to any other it follows by this argument that all the variables are equal to v_a . Hence the only solutions to the equations are $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. There are α_0 variables. Hence the number of the α_1 equations which are linearly independent must be $\alpha_0 - 1$.

Hence the rank of the matrix A is $\alpha_0 - 1$.

From these equations, Veblen stated that, as there are α_0 equations (vertices) among α_1 variables (edges), and $\alpha_2 - 1$ solutions, the rank of the matrix A is $\alpha_0 - 1$. This gives:

$$\alpha_1 - (\alpha_0 - 1) = \alpha_2 - 1, \text{ or } \alpha_0 - \alpha_1 + \alpha_2 = 2,$$

which is Euler’s formula.

A third set of equations (3) is taken from the rows of B and, as Veblen stated, they are ‘entirely analogous to the equations (2)’; this gives the rank of the matrix B as $\alpha_2 - 1$. The fourth set of equations (4) is derived from the columns of B . The solutions of these equations are given in the rows of A , with $\alpha_0 - 1$ rows linearly independent. This result can also be found from Euler’s theorem, since the rank of the matrix B is $\alpha_2 - 1$ and the number of linearly independent solutions is $\alpha_1 - \alpha_2 + 1$.

Having established his matrices, Veblen turned to the four-colour problem. Using a field of four elements $(0, 1, i, i + 1)$ to represent the four colours, he defined two elements α and β of the field to be equal if and only if $\alpha + \beta = 0 \pmod{2}$. It follows that if a set of values $(y_1, y_2, \dots, y_{\alpha_2})$ can be found that satisfies none of the equations (3), corresponding to the rows of the matrix B , then this provides a solution to the four-colour problem.

Veblen continued by saying that the:

... set of values $(y_1, y_2, \dots, y_{a_2})$ may be regarded as a point in a finite projective space of $a_2 - 1$ dimensions provided we exclude the set $(0, 0, 0, \dots, 0)$.

He then explored different spaces and reformulated the four-colour problem in terms of subspaces of a projective space to the following statement:

In a finite projective space of (a_2-1) dimensions with three points on a line there are a certain number of spaces S^{a_2-n} of dimensionality a_2-n , one for each odd cycle C_n . They all have one point in common. The map can be colored in the four colors if and only if there exists a point not on any of these S^{a_2-n} 's. There are as many distinct ways of coloring the maps (aside from permutations of the colors) as there are real lines in the (a_2-1) -space which do not meet any S^{a_2-n} (n odd).

The paper then investigated further solutions to the sets of equations that provided conditions under which the four-colour problem could be solved. It concluded by pointing out that the generated equations were essentially the same equations of congruence defined by Heawood in his 1898 paper. Veblen's final paragraph, which summed up the paper's content, reads:

To solve the four-color problem it is necessary and sufficient to find a solution of these equations in which none of the variables vanish. The variables may be interpreted as coordinates of points in a finite projective space of a_0 -dimensions in which there are four points on every line.

The paper may not have contained new steps upon which the eventual solution of the four-colour theorem was built, but it gave a reformation of the problem, generalising the 'system of Heawood congruences' [8].

4.3 *The Cambridge Colloquium 1916: Analysis Situs*

This publication was the collected addresses presented by Veblen to the American Mathematical Society as their Colloquium Lectures in 1916. The following paragraph from the *Preface* succinctly sets the scene for the book's content and indicates the modest and unassuming personality of its author:

The Cambridge Colloquium lectures on Analysis Situs were intended as an introduction to the problem of discovering the n -dimensional manifolds and characterizing them by means of invariants. For the present publication the material of the lectures has been thoroughly revised and is presented in a more formal way. It thus constitutes something like a systematic treatise on the elements of Analysis Situs. The author does not, however, imagine that it is in any sense a definitive treatment. For the subject is still in such a state that the best welcome which can be offered to any comprehensive treatment is to wish it a speedy obsolescence.

Also in the preface, Veblen recorded his indebtedness to Dr Philip Franklin, who ‘assisted with ... the manuscript, the drawings, and the proof-sheets’. These papers, when later published as a book, constituted Veblen’s most important work on graph theory and became a standard reference work on topology for many years.

The publication was divided into five chapters, the first titled *Linear Graphs*. At the start of this first chapter Veblen established his *Fundamental Definitions*, illustrated using K_4 , the graph used in his earlier paper (but with a different notation) and shown earlier in this thesis. He built on the following basic definitions to cover the then-known parameters and formulas for linear graphs:

- 0-dimensional simplex = single point
- 1-dimensional simplex = segment or edge
- 0-dimensional cell (0-cell) = end or vertex
- 1-dimensional cell (1-cell) = the points of a segment
- 0-dimensional complex = the set of distinct 0-cells
- 1-dimensional complex = linear graph

Veblen assembled Kirchhoff’s ideas, as developed by Poincaré, to define the number of cycles in a fundamental set, a value related to the rank of the incidence matrix. Veblen went on to outline how Kirchhoff developed a fundamental set of cycles related to a spanning tree within a graph. He also detailed a way to show that every connected graph contains a spanning tree.

Veblen developed incidence matrices for these multidimensional structures and included a discussion of the theory of the n -cell, regular complexes, manifolds, and the dual complexes associated with them. He also included a description of Betti numbers. The Betti number is the nullity N of a graph G , and relates the number of vertices V , edges E , and faces F , by a development of Euler's polyhedron formula given as:

$$N = E - V + F.$$

Veblen's contribution to graph theory was not just through the work he published, but also by his direct influence on other mathematicians such as Birkhoff and Franklin. Birkhoff, followed closely behind Veblen chronologically and their academic lives were closely linked, not just through their mathematical work, but also in the development of American scholarship and its standing in the world. As Franklin's postgraduate supervisor, Veblen was undoubtedly directly instrumental in Franklin's thesis subject and for his continuing interest in graph theory and the four-colour problem (see Chapter 5).

4.4 George David Birkhoff (1884–1944)

George David Birkhoff was a complex man. He was arguably the foremost American mathematician of the 1920s and 1930s. None other than Albert Einstein accused him as being 'one of the world's great anti-Semites' [9], but he was warmly praised for the encouragement and support he gave his research students. He was known not to support women in academia, but was described as 'intensely social'. Despite these contradictions, he was highly respected in his homeland and around the world.

Birkhoff attended the University of Chicago in 1902, where he met Veblen, then studied at Harvard University from 1903 to 1905, being awarded an AB in 1905 and an AM in 1906. He returned to the University of Chicago in 1905 for postgraduate study under E H Moore, and was awarded a PhD degree in 1907 for his thesis *Asymptotic*

Properties of Certain Ordinary Differential Equations with Applications to Boundary Value and Expansion Problems, the work being heavily influenced by the mathematics of Poincaré. Subsequent to being awarded his doctorate, Birkhoff took up an appointment in 1907 as an instructor of mathematics at the University of Wisconsin. In 1909, he moved to Princeton University as a preceptor in mathematics, and in 1911 he was appointed a professor where he worked mainly on dynamics and mathematical physics. In 1912, he returned to Harvard as an assistant professor and became a full professor there in 1919, remaining at Harvard for the rest of his life.

Birkhoff's first contribution to the story of graph theory in America was his 1912 paper entitled *A determinant formula for the number of ways of coloring a map* [10], in which he introduced chromatic polynomials. His second was his 1913 pioneering paper, *The reducibility of maps* [11], which would prove to be significant in the development of the solution to the four-colour problem. His interest and subsequent mild obsession with the four-colour conjecture was triggered by his attending Veblen's seminar in *analysis situs* when they were together at Princeton [12]. His son Garrett also became a prominent mathematician, and in later life recorded that his mother whilst on honeymoon was requested by her new husband to prepare complicated maps for him to colour [13]. In later life [13], the older Birkhoff was to rue the effort he had expended on the problem, even though it had been a keen ambition of his to provide a solution — then again, perhaps it was because his ambition was not realised. In 1935, Hassler Whitney (see Chapter 6) wrote of the work [14]:

A major step in advance was given by G.D. Birkhoff ... In the early 1930s, when I was at Harvard, exploring the problem among other things, Birkhoff told me that every great mathematician had studied the problem, and thought at some time that he had proved the theorem (I took it that Birkhoff included himself here).

In 1972, Thomas L Saaty (1926) honoured Birkhoff's quest for a solution to the four-colour problem by defining the *Birkhoff number* which he defined to be the minimum

number of countries that a map must have if it cannot be coloured admissibly with four colours. However as the four-colour problem has been solved the definition needs to be revised. In their book *The Four-Color Problem* Rudolf and Gerda Fritsch [15] state:

The Birkhoff number is b on day t if on day t it has been shown that a map that cannot be admissibly colored with four colors must contain at least b countries. For the years 1852 to 1879, by the Weiske theorem (Theorem 4.5.1), the Birkhoff number was 6 (Corollary 4.5.2). In 1879, by way of Kempe's deliberations, the Birkhoff number had risen to 13.

The search for reducible configurations had begun soon after the conjecture had been posed in 1852. It continued until the solution was provided in 1976 by K Appel and W Haken (see Chapter 9) whose proof of the four-colour theorem implies that the Birkhoff number is infinite. Before then, the Birkhoff number (based on [15]) was:

<u>Name</u>	<u>Date t</u>	<u>Birkhoff Number b</u>
B G Weiske [15]	before 1852*	6
A B Kempe [16]	1879	13
P Franklin [17]	1922	26
C N Reynolds [18]	1926	28
P Franklin [19]	1938	32
C E Winn [20]	1940	36
O Ore and J G Stemple [21]	1970	41
W R Stromquist [22]	1973	45
J Mayer [23]	1974	48
W R Stromquist [24]	1974	52
J Mayer [25]	1975	96

* 'For the years 1852 to 1879, by the Weiske theorem, the Birkhoff number was 6'. [15].

The Weiske theorem is 'There exists no map with five pairwise neighbouring countries'

Also in 1913, Birkhoff proved Henri Poincaré's *Last Geometric Theorem*, a special case of the three-body problem, which Poincaré had posed in 1912 but had been unable to solve. This work was primarily the reason for Birkhoff's international fame, and he was regarded, at that time, as North America's leading mathematician; his proof is still widely

used today. Marston Moore said that ‘Poincaré was Birkhoff’s true teacher’ [12]; indeed, few scholars knew and understood Poincaré’s work better than Birkhoff. Scholars in Göttingen were somewhat aggrieved and surprised that an American had solved this problem. After working on other areas of mathematics Birkhoff later returned to the problem of colouring maps and in 1930 published a paper entitled *On the number of ways of coloring a map* [26], also on chromatic polynomials.

The American Mathematical Society celebrated its fiftieth anniversary in 1938 and invited Birkhoff to give the main address, on fifty years of American mathematics. His paper [27] included not only a proud claim of the standing and quantity of ‘highly creative ... American mathematicians’, but also a warning regarding the influx into the USA of scholars from the oppression being experienced in Europe. Birkhoff believed that young American mathematicians would be deprived of opportunities for higher positions and research posts in the University system, that they would be restricted to what he described as ‘hewers of wood and drawers of water’ — that is, employed — provided that they could obtain a position against increasing competition as full-time teachers at high schools and colleges, but without the chance to carry out research. In this matter, Birkhoff was wrong: the very people about whom he was concerned became leaders in American mathematics and helped the USA to become the leading mathematical country in the world.

In 1939, Birkhoff became involved in correspondence with a Cleveland newspaper, which (he believed) had published the fact that he had solved the four-colour conjecture. Perhaps it was a badly worded article [28] and intended not to claim that Birkhoff had found a proof, but Birkhoff believed that somewhere in his work on chromatic polynomials there contained a way to the answer. The article was sufficiently ambiguous that Birkhoff felt it necessary to write to the newspaper. The reply [29] that it solicited did not contain an apology, but said:

I don’t think that my original article gave the impression that you had solved the problem, although it did say that you were making progress.

Birkhoff saw the potential of calculating devices as aids to mathematicians and, using funds from a bequest to Harvard and considerable assistance from IBM financed the building, by Howard Aiken, of a large calculator at Harvard that was in operation before the end of World War II. Because of the success of this machine, the U. S. Navy ordered three more of improved specification for use in naval laboratories.

Birkhoff often appeared somewhat detached from the goings-on in the world around him, including in his attitude towards politics and social intercourse. This sometimes gave rise to misunderstandings and hostility. His approach to discussion was one of brevity and simplicity without explanation. He was also able to appreciate and enunciate opposing views of a problem, which on occasion led to a hearer obtaining an incorrect impression of Birkhoff's own opinion. It was sometimes difficult to understand where Birkhoff stood on a particular subject [12].

During World War II Birkhoff and his wife were good-will ambassadors, travelling to South America and Mexico, and cooperating in Nelson Rockefeller's effort to promote solidarity against Hitler.

4.5 A determinant formula for the number of ways of coloring a map [10]

Just one week after Veblen gave his paper on modular equations, his Princeton colleague Birkhoff published the first of his major papers on the four-colour problem. Like other scholars who worked on the problem, he was unable to provide a solution, but did suggest a new avenue of investigation, a quantitative approach; this was the development of a polynomial $P(\lambda)$ to count the number of ways of colouring the regions of a map M using λ colours.

His paper included two examples. The first was of a map M (Figure 1) with three mutually adjacent regions (ignoring the outside region), where region A can be coloured

with any of the λ colours; region B by any of the remaining $\lambda - 1$ colours; and region C by any of the remaining $\lambda - 2$ colours. This gives the total number of ways of colouring M as

$$P(\lambda) = \lambda(\lambda - 1)(\lambda - 2),$$

so if only three colours are available, then the number of ways of colouring M is:

$$P(3) = 3(3 - 1)(3 - 2) = 6.$$

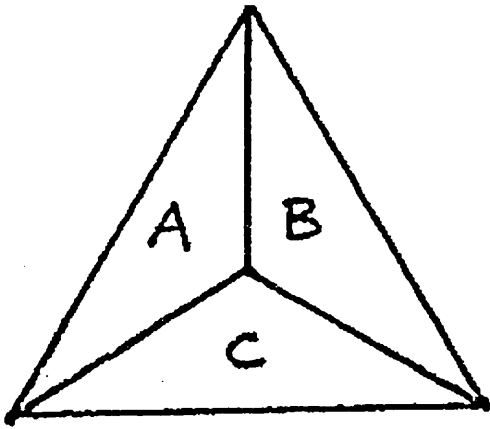


Figure 1

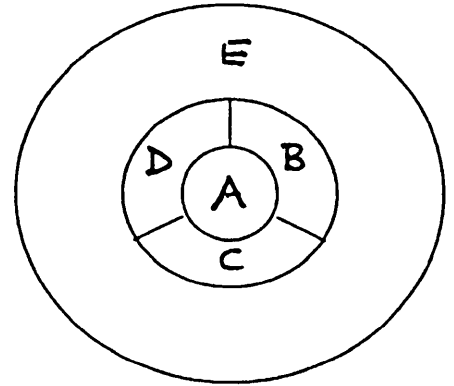


Figure 2

His second example was of a map of five regions (Figure 2). Here region A can be coloured with any of the λ colours available; B with $\lambda - 1$ colours; C with $\lambda - 2$ colours; D with $\lambda - 3$ colours; and E also with $\lambda - 3$ colours, so:

$$P(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^2,$$

so for $\lambda = 4$ $P(4) = 4(4 - 1)(4 - 2)(4 - 3)^2 = 24.$

The above equations were also calculated from his general formula:

$$P(\lambda) = \sum_{i=1}^n \lambda^i \sum_{k=0}^{n-i} (-1)^k (i, k),$$

where n is the number of regions of M and (i, k) denotes ‘the number of ways of *reducing down* the map M to a submap of i regions by k simple or multiple coalescences’. The boundary conditions given were $(i, k) = 0$ for $k > n - i$; $(n, 0) = 1$; and $(i, 0) = 0$ for $i < n$. Birkhoff found that the function P is always a polynomial in λ and provided a proof of the general formula for the coefficients of $P(\lambda)$ which was somewhat complex and twenty

years later Hassler Whitney provided a simpler procedure (see Chapter 6). Birkhoff also included the warning that ‘the value of (i, k) is not immediately obtained’ and indicated that ‘more complicated maps would require ... considerable computation ... to determine $P(\lambda)$ from the above equation or by inspection of the map itself’.

His main objective was to show that $P(4) > 0$, but he was unable to achieve this. However, many properties of $P(\lambda)$ were obtained and the paper was the first solid step in the use of a quantitative approach to the four-colour problem. His polynomial function is now known as the *chromatic polynomial* of the map.

Birkhoff had hoped that his theory of chromatic polynomials could be built upon and that analytic function theory methods could be utilised to achieve a solution to the four-colour problem. A lengthy paper [30], written jointly by Birkhoff and Daniel Clark Lewis, Jr. (see Chapter 8) but published in 1946 after Birkhoff’s death, attempted to collect into one document a significant amount of quantitative work on the colouring of maps and graphs.

4.6 Reducibility

Birkhoff’s pioneering paper *The reducibility of maps* [11] was published in the *American Journal of Mathematics* in 1913. In the introduction to the paper, Birkhoff maintained that all previous work on the four-colour problem could be stated as in terms of four ‘reductions’ (restrictive conditions), which he stated as:

If more than three boundary lines meet at any vertex of a map, the coloring of the map may be reduced to the coloring of a map of fewer regions.

If any region of a map is multiply-connected, the coloring of the map may be reduced to the coloring of maps of fewer regions.

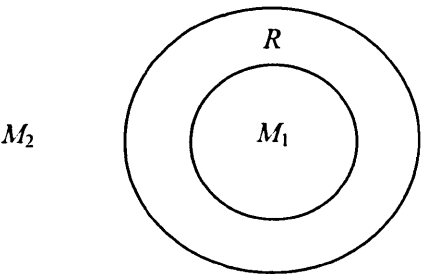
If two or three regions of a map form a multiply-connected region, the coloring of the map may be reduced to the coloring of maps of fewer regions.

If the map contains any 1-, 2-, 3- or 4-sided region, the coloring of the map may be reduced to the coloring of a map of fewer regions.

The introduction concluded with the purpose of the paper: ‘to show that there exists a number of further reductions which may be effected with the aid of the notion of chains due to A B Kempe’.

Earlier, in 1904, Wernicke had defined some unavoidable sets; he proved that a cubic map that contains no digon, triangle, or square must contain either two adjacent pentagons or a pentagon adjacent to a hexagon.

In his paper, Birkhoff employed a qualitative approach, developing Kempe’s arguments by looking at rings of regions to find reducible configurations; that is an arrangement of countries that cannot occur in a map comprising a number of countries that cannot be coloured with four (or any given number of) colours, while a map with fewer countries can be coloured. He referred to the diagram below:



Birkhoff considered a ring R of regions that divided a map M into two sets of regions M_1 and M_2 that collectively made $M = M_1 + M_2 + R$. The colouring of the maps $M_1 + R$ and $M_2 + R$ can be combined to give a complete colouring of M providing that the two maps have the same colouring on R . Birkhoff applied the method of Kempe-chains to four colourings of $M_1 + R$ and $M_2 + R$ and proved that an irreducible map can contain no ring of four regions. He also considered rings of five and six regions surrounding a given configuration, showing that for five rings the configuration is reducible except where the ring encloses a pentagon. He had less success with rings of six regions.

Included in his paper was a list of three possible outcomes of the four-colour problem. Two of these possibilities stated that ‘All maps can be colored in four colors ... with maps either including reducible rings or more complex reductions.’ The third possibility given was that ‘There exist maps which can not be colored in four colors’, thereby showing a little caution. However, his logical evaluation of possible reducible configurations provided the basis upon which future mathematicians would continue the work. One of these was Philip Franklin (see Chapter 5). Many of these scholars would use Birkhoff’s theorem on rings of five regions.

4.7 On the number of ways of colouring a map

In this 1930 paper [26], Birkhoff returned to the subject of chromatic polynomials and as with his 1912 paper on chromatic polynomials, he employed the quantitative approach to the problem of colouring maps. The paper contains a proof of the theorem:

If $P_n(\lambda)$ denotes the number of ways of colouring any map of n regions on the sphere in λ (or fewer) colors, then for $n \geq 3$ and $\lambda \neq 4$

$$\text{Then } P_n(\lambda) \geq \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-3}. \quad (1)$$

Had he been able to include $\lambda = 4$ in this inequality, he would have proved the four colour theorem.

He investigated those maps which had a single central region enclosed by rings with increasing numbers of regions. He explained his approach as:

Consider a map M [of n regions] ... If there is a region a of M which is multiply connected, form a new map M_1 by drawing together two boundary lines, one on each side of this region till they touch at a point, forming a new vertex. Continue this process until a map N still of n regions but with none of its regions multiply connected, is obtained. Since the pairs of regions having a common boundary are the same for N as for M , any colouring of N furnishes a colouring for M . Hence we need only to prove that (1) holds for maps N in which there are no multiply connected regions.

He then determined all possible ways to colour the resulting map.

From these colourings, he developed polynomials in λ , which resulted in the above inequality. He concluded the paper with the statement:

It is deserving of remark that the inequality (1) of the theorem is the best equality of its type, i.e. for every pair of numbers $\lambda > 4$ and $n \geq 3$ there is a map M in which

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-3}$$

Throughout his life Birkhoff carried out a considerable amount of work on chromatic polynomials. He always hoped that they would lead to a solution to the four-colour problem, but that was not to be. We return to Birkhoff, and his monumental paper of 1946 [30], in Chapter 8.

Conclusion

In the quest for a solution to the four-colour problem, little of major significance was accomplished for some time after Birkhoff's initial papers on the subject were published. However, from the 1920s, some modest but positive contributions were made in America, following the work of Veblen and Birkhoff. These were by, amongst others, Franklin, Clarence N Reynolds, C E Winn, and Hassler Whitney (see Chapters 5 and 6). A solution of the four-colour problem had to wait until the 1970s before a major breakthrough was made, but that is outside the scope of this narrative; however there was still a considerable amount of work done in America in that period in the field of graph theory.

Before progressing further with the history of graph theory in the United States of America, the following considers the effects on mathematics in that country by the Great War and the depression.

4.8 The First World War

When the USA declared war on Germany on 6 April 1917, it initiated a programme of deployment of academics from universities and colleges across the country to assist in the war effort. Included in that deployment were nearly two hundred mathematicians who, in various capacities, became engaged in war service during World War I. Indeed, some were to make considerable contributions again in World War II, and to have a significant influence on the participation of mathematicians in the second global conflict.

Of the mathematicians called to arms, some saw active service with the army, navy or air force, while others acted as instructors and technical administrators. However, the majority were employed in research into war-related technologies. While most were stationed in North America, some served in France, Italy and Britain. Included among them were three who feature in this thesis:

- E H Moore, aged 55 in 1917, and head of the mathematics department at the University of Chicago, became Chairman of the *National Research Council* (representing mathematics), although he remained in Chicago.

- Oswald Veblen, aged 37 and Professor of Mathematics at Princeton University, was commissioned Major in the US army and assigned to research in ordnance at the Aberdeen Proving Ground, Maryland.

- Philip Franklin, aged 19 and an undergraduate at the College of the City of New York, was to work at the Aberdeen Proving Grounds alongside Veblen.

No mentions of G D Birkhoff in connection with the patriotic contribution of American mathematicians during the Great War were found during the research for this thesis. In particular, his name is not included in the list *American Mathematicians in War Service* published by the American Mathematical Society [31]. Additionally no mention of Wernicke in relation to war service has been found but it is possible that he had not become a US citizen by the time of America's entry into the war; although whilst he was

Professor of Modern Languages at the University of Kentucky in the years 1894–1906 ‘he was in charge of the University’s military training, holding a commission as a colonel in the Kentucky militia’ [13].

This involvement of academics in the practical world of war-related technologies meant that many mathematicians used only to the *pure* variety became exposed to more *applied* topics. This had a two-fold benefit; applied mathematics became more highly regarded, and some great mathematical minds saw opportunities for significant research in that area.

After the war, life returned to normal for these mathematicians. Most returned to their academic positions, while some, like Franklin, completed their degrees. Life was later disrupted again with the stock market collapse in 1929, which led to a worldwide economic depression and hard times for all. During the Great Depression, there were many business failures, including banks, leading to high levels of unemployment and privation for a substantial part of the population. At times, unemployment ran at 25 per cent, at a time of no unemployment benefit and little in the way of pension plans.

The effects of this global disaster aided Adolf Hitler in becoming Chancellor of Germany in 1933. The economy of Germany, still recovering from the effects of the Great War, was in a sorry state and the German people were experiencing a poor existence. The Wall Street crash only worsened their plight, so many took the easy option and voted overwhelmingly for someone who promised to improve their lot. A considerable majority of the German people elected Hitler and thereby opening the door to another global conflict.

In America, as in other countries, the depression affected most academics although less so for full professors with their tenure of position and with salaries reasonably protected. Although their remuneration was subject to reductions of between 10 to 15 per cent around 1932 or 1933, this coincided with a general fall of costs and prices, so those at

the higher end of the academic ladder remained relatively well off. This was not the case for those on the lower rungs, or those with recently acquired degrees: in the 1930s, too many with mathematics doctorates were chasing too few jobs. Both state and central governments drastically reduced their funding of establishments of higher education, while many colleges and universities were forced to terminate the contracts of some junior staff. Research grants became rare, and economies were made in the areas of travel and accommodation for attending meetings and conferences, sabbaticals and secretarial assistance. Added to this were instances of increased teaching workloads to cover for released staff [32].

In the 1930s, many mathematicians with doctorates were unemployed, whilst others were engaged in positions such as high-school teaching or industry that were not closely related to their research (or, indeed, to mathematics). In 1933 a commission set up by the Mathematical Association of America reported that between forty and fifty (out of around 150) new PhDs in mathematics had not found satisfactory employment by the beginning of October 1934.

Even forsaking colleges and universities and attempting to obtain a position at a high school was not without its problems. Many heads of high-school mathematics departments were wary of bringing in those who were better qualified and had greater mathematical knowledge and understanding than themselves, and who may not teach well. Additionally, the secretary of the American Mathematical Society, R D G Richardson, reported that seventy-five scholars with PhDs in mathematics were unemployed in 1935 [32]. For Jewish postgraduates the situation was already being made worse by the prevalent anti-Semitism in America, particularly at some of the top universities. Even if a mathematician were fortunate enough to obtain a teaching position at a college, university or high school, it was unlikely that an increase in salary would be awarded unless it was attached to a promotion that could take some years to achieve.

This situation had a number of benefits. One was that with the employment of highly qualified mathematicians, the potential level of teaching at high schools could possibly be raised. Another was that, as in many institutions at that time, faculty members did not give talks outside of their classroom duties, attend seminars or learned meetings, or publish papers; but with all the young new PhDs and the shortage of jobs came severe competition that resulted in an increased number of publications. Another result of the Depression was that some scholars were lost to mathematics, most going into industry, and many more never reached their full mathematical potential. The whole situation, further aggravated in the latter 1930s by the influx of mathematicians from mainly German-speaking countries, continued until the USA declared war on Japan and Germany in December 1941.

Although the effects of the Great War and the Depression were not unique to America, the situation was a brake to the academic development of that country. This was particularly regrettable, as America had really begun to put in place a higher education programme and a research culture only around the beginning of the century. Nevertheless, the USA had enough remarkable and resolute mathematicians to continue their increase in standing around the world.

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Chapter 5

American graph theory between the wars

In the second decade of the twentieth century, some US mathematicians followed the example of Veblen and Birkhoff. One, a postgraduate student of Veblen, was Philip Franklin whose doctoral thesis subject was on the four-colour problem. Others were Henry Roy Brahana, Clarence Reynolds, C E Winn, and Hassler Whitney (see Chapter 6).

5.1 Philip Franklin (1898–1965)

Philip Franklin was a quiet unassuming man who seemed less able or willing than others to promote himself, but who nevertheless provided undemonstrative encouragement and support to his students. He was loyal to his colleagues and to the faculties to which he was appointed. Because he preferred to maintain his privacy, there is less biographical information available for him than of other mathematicians in this thesis.

In 1914, he enrolled at the College of the City of New York and graduated with a BS degree in 1918. He undertook postgraduate study at Princeton University where he received an MA degree in 1920 and his PhD in 1921. The influence of his supervisor, Oswald Veblen, is indicated by his choice of research subject, the four-colour problem. It was a time of increasing interest in graph theory and colouring problems in America.

As mentioned earlier, the First World War interrupted Franklin's undergraduate studies, when he spent time in the range-firing section of the Army's Ordnance Proving Grounds at Aberdeen. Whilst there, he met another member of Veblen's group, Norbert Wiener (1894–1964), the inventor of cybernetics. A child prodigy, Wiener enrolled in college at age 11 and obtained his PhD from Harvard at age 18. Wiener believed that Harvard's newly appointed assistant professor, G D Birkhoff, was partly responsible for his not getting an appointment claiming that Birkhoff showed him a special antipathy, as a

Jew and, ultimately, as a possible rival. Birkhoff was to cross paths with Wiener on two more occasions at least, and it can be surmised that Birkhoff's anti-Semitism and fear of a possible challenger played no small part in the encounters. Birkhoff was one of a small number of Harvard mathematicians in the 1920s who warned Wiener off pursuing the then-topical subject of potential theory so as to leave the field clear for other Harvard selected scholars. He was also involved in Wiener's being badly treated during an invited year-long stay in Göttingen on a Guggenheim grant, a situation that drove Wiener to depression and the verge of a nervous breakdown. Wiener eventually found a position in the mathematics faculty at the Massachusetts Institute of Technology, which he entered in 1919 and remained there for forty-five years until his death.

Wiener was to become Franklin's brother-in-law, as Franklin married Norbert's sister Constance who was a mathematician in her own right. During their war service, a fellow mathematician was David V Widder, who had a bunk in the same barracks as Franklin and Wiener and recalled [1]:

I learned a lot from these enthusiasts, but at times they inhibited sleep when they talked mathematics far into the night. On one occasion I hid the light bulb, hoping to induce earlier quiet.

After the war and returned to academic life Franklin presented some of his researches to the National Academy of Sciences on 17 November 1920. In his full thesis, he showed that each of the configurations listed below is reducible:

- a pentagon in contact with three pentagons and one hexagon
- a pentagon surrounded by two pentagons and three hexagons
- a hexagon surrounded by four pentagons and two hexagons
- any n -sided polygon in contact with $n - 1$ pentagons

Franklin remained at Princeton as an instructor in mathematics for a year after gaining his doctorate, and then spent two years at Harvard University as Benjamin Peirce Instructor. In 1924, he moved to the Massachusetts Institute of Technology as an instructor in mathematics, becoming an assistant professor in 1925 and an associate professor in

1930. He became a full professor in 1937, a position he held until his retirement in 1964. Even then, he continued to give classes there on a part-time basis. Franklin was influential in the introduction of graph theory to MIT as recorded by Dirk J. Struik [2]:

Since Franklin brought topology to MIT in his “analysis situs” form, and Wiener in his “point-set-Lebesgue” form, we see that it came to the Institute through two brothers-in-law.

Franklin published the following five papers on graph theory:

- 1 *The four color problem* (1922)
- 2 *The electric currents in a network* (1925)
- 3 *A six color problem* (1934)
- 4 *Note on the four color problem* (1938)
- 5 *The four color problem* (1941)

5.2 Franklin’s graph theory papers

Paper 1 *The four color problem*

Following the appearance of his PhD thesis the previous year this paper [3] was published in 1922. In it Franklin made use of Kempe chains and developed the work of Wernicke and Birkhoff on reducible configurations. This paper was a significant contribution to the idea of reducibility as an avenue towards solving the four-colour problem. It was the first in a continuing line of papers using this approach over the years. Franklin’s development of the subject was a model of clarity and precision.

The *counting formula*, which is an application of Euler’s polyhedron formula (similar to that used by Kempe in his 1879 paper – see Chapter 2), that details the numbers of each k -sided country in a cubic map is:

if C_k is the number of k -sided countries in a cubic map, then

$$4C_2 + 3C_3 + 2C_4 + C_5 - C_7 - 2C_8 - 3C_9 - 4C_{10} - \dots = 12.$$

Franklin used this formula to deduce the important result that every cubic graph must contain at least one digon, triangle, square, or one of the following:

- a pentagon adjacent to two other pentagons
- a pentagon adjacent to a pentagon and a hexagon
- a pentagon adjacent to two hexagons

This gives rise to an unavoidable set with nine configurations. Additionally, he proved that no 6-vertex region can have three consecutive 5-vertices regions as neighbours.

Franklin then introduced new reducible configurations including; a pentagon in contact with three pentagons and a hexagon; a pentagon surrounded by two pentagons and three hexagons; and a hexagon surrounded by four pentagons and two hexagons; and any n -sided polygon in contact with $n - 1$ pentagons. He then deduced the theorem:

Every map containing 25 or fewer regions can be colored in four colors.

This was the first increase in the Birkhoff number since 1879 (see Chapter 4), when Kempe proved that maps of up to twelve regions can be coloured with four colours. Franklin continued:

The question naturally arises whether 25 is the greatest number for which we can prove such a theorem as the above on the basis of the reductions already described.

He was unable to answer this question but went on to say:

We exhibit a map of 42 regions which satisfies all the properties of irreducible maps given by previous writers as well as those derived in this paper

His example can be coloured in four colours. This paper was an encouraging step along the way that others were to follow.

Paper 2 *The electric currents in a network*

Franklin's second graph theory paper [4] had nothing to do with map colouring. It gave a shorter and alternative proof to that of Kirchhoff for the calculations of electric currents in a circuit in terms of voltages and resistances. As Franklin pointed out, Kirchhoff's paper 'was

essentially the first contribution to the study of analysis situs of a linear graph'. Additionally, Franklin's proof intended, to 'also bring to light the mathematical nature of the result'. It was presented to the American Mathematical Society on 25 October 1924. In it he credited Veblen's *Cambridge Colloquium Lectures on Analysis Situs* for the terminology and treatment of linear graphs, using as his example the graph K_4 and its corresponding matrix that Veblen had included in his own paper. Franklin's approach was to consider an electrical network as an oriented linear graph.

Paper 3 *A six color problem*

This paper [5] was published in the 1933–34 issue of the Massachusetts Institute of Technology *Journal* and in it he showed that Heawood's bound for the Klein bottle was not attained. The *Introduction*, following considerations of Kempe, Heawood and Heffter, begins:

The problem of coloring a particular map in such a way that any two countries contiguous along an edge have distinct colors leads to a more general question. For any surface, what is the smallest number of colors sufficient to color any map drawn on it? For surfaces topologically equivalent to the plane or sphere it is known that four colors are sometimes necessary, and that five will always suffice. Whether five are ever actually needed is an open question.

This paper was presented to the American Mathematical Society and was published in their *Journal* in 1934. In it he reviewed Heawood's formula for the number of colours sufficient for any map on any surface, and pointed out that for the 1-sided (non-orientable) surface of characteristic 0 (the Klein bottle), the Heawood number is 7. However Franklin developed an argument that resulted in the two following formulae:

Theorem I. Every map on a manifold with $K = 0$, except that of seven mutually contiguous hexagons, is reducible with respect to six colors.

Theorem II. Every map on a Klein sack is colourable in six colors.

Franklin further stated that as Heawood's 'formula is incorrect for this surface' then it 'may also fail in other cases'.

Franklin discussed a closed surface (2-dimensional manifold) of characteristic K such that the number of regions (a_2), edges (a_1), and vertices (a_0) are related by:

$$a_0 - a_1 + a_2 = K.$$

Paper 4 *Note on the four color problem*

In this paper [6], published in 1938 and read to the American Mathematical Society, Franklin extended the Birkhoff number to 31 stating:

I shall prove that every map on a sphere containing not more than 31 regions can be colored.

After introducing equations previously developed by Kempe, Birkhoff, Euler, Reynolds (see later), and himself (in his paper *The four color problem*), he postulated that any irreducible map must contain at least 32 regions. This was based on his equation

$$a = 2 + v/2$$

which he derived from work by Kempe, Birkhoff and Euler, where a is the number of regions and v is the number of vertices; additionally he stated that “all vertices are triple and all polygons have at least five sides”. Since a is an integer, Franklin hypothesised that if it could be shown that for an irreducible map v must exceed 58, then $a \geq 32$, and thereby satisfies his assumption. To achieve this Franklin explored a number of configurations of maps, namely:

- exactly 6 heptagons and no other polygons with more than 7 sides
- more than 6 heptagons and no other polygons with more than 7 sides
- 1 octagon, 5 or more heptagons and no other polygons with more than 7 sides
- either 1 polygon with more than 8 sides, or at most 2 polygons with more than 7 sides
- 3 or more polygons with more than 7 sides

In each case Franklin showed that the number of vertices in all possible configurations considered is greater than 58, which confirmed that every irreducible map must contain at

least 32 regions, and therefore every map on a sphere with at most 31 regions is colourable with four colours.

The final three sections of the paper contained details of four individual reducible configurations and a generalisation of a result of Heawood's in his 1936 paper *Failures in congruences connected with the four-colour map theorem* (see Chapter 3).

Paper 5 *The four color problem*

This paper [7] was a revised version of a lecture on the four-colour problem that Franklin gave at the Galois Institute of Mathematics at Long Island University, New York in 1939. It was an expository paper and as was usual from Franklin, the paper was well written and lucid.

His introduction described the topics he intended to cover, providing an appropriate historical setting and continued:

What I propose to do here is to prove the result concerning five colors and illustrate how the problem may be simplified to the consideration of regular maps. I shall then define the notion of reducible configuration, establish some of the simplest of these, and describe several others.

He defined regular maps as:

The vertex of a map is called a triple vertex if three and only three edges meet there. A map all of whose vertices are triple, and all of whose regions are simply connected, is called a regular map.

He went on to say that he would mention some alternative problems that were equivalent to the four-colour problem, or that would imply that four colours would be sufficient to colour maps. He also stated that he would introduce 'specialized' and generalised theorems including some on 2, 3, 5, 6 and 7 colours.

He introduced reducibility and stated that if five colours are available then regions of two, three, and four sides are reducible. Then using the fact that every regular map must contain at least one region of fewer than six sides, he concluded 'that every such map is

reducible in five colors'. Then by contradiction, Franklin proved that every map on a sphere can be coloured in five colours.

The paper then investigated reducible configurations when only four colours are available and summarised possibilities, giving credit to the originators:

Among the simpler reducible configurations are non-triple vertices, multiple connected regions, and rings of four or fewer regions (Birkhoff). Some other known reducible configurations are:

(a) A ring of five regions not surrounding a single pentagon (Birkhoff). This is fundamental in the proof of most of the other reductions.

(b) A pentagon adjacent to three consecutive pentagons (Birkhoff).

(c) A hexagon adjacent to three consecutive pentagons (the writer).

(d) A heptagon adjacent to four consecutive pentagons (Winn).

(e) A ring formed of an even number of pentagons and one or two adjacent additional regions (Winn).

(f) A ring formed of an even number of hexagons and zero or more pairs of adjacent pentagons surrounding one, two, or three contiguous regions (Errera).

The reductions (e) and (f) originated with Birkhoff for rings surrounding a single region, without the additional regions or pairs of pentagons. The extension to them was due to the writer. Later Errera gave the extension for both types indicated in (f), and with certain restrictions, to any such rings. Finally Winn removed the restrictions for (e).

In Section 8 Franklin went on to discuss *Special Coloration Theorems* saying:

Several interesting results concerning special coloration have been proved.

One was the two-colour theorem:

A necessary and sufficient condition for a map to be coloured in two colours is that all the vertices be even.

A chessboard map is an example. Another special case is the three-colour theorem:

A necessary and sufficient condition for a regular map to be coloured in at most three colours is that each region has an even number of sides.

These theorems were previously formulated by Kempe.

He then discussed in, *Special Classes of Colorable Maps*, classes of maps which can be coloured in not more than four colours. Stating the theorem:

A regular map with each region of $3n$ sides can be coloured in four colours.

giving as an example the icosahedrons; he also gave a result of a less restrictive character:

A regular map containing at most one region of more than six sides can be coloured

This was proved by Winn (see later). Franklin continued with *Methods Involving Vertices and Edges*, developing Tait's argument that the colouring of the regions of a regular map in four colours is equivalent to the colouring of its edges in three colours. There followed a review of Tait's conjecture and Petersen's theorem (see Chapter 3), together with results of Kuratowski, Frink, Hamilton, and Whitney.

Franklin then looked at the problem of colouring volumes in space, noting, as had Frederick Guthrie in 1880, that the number of colours required remains unlimited. In later sections of the paper he explored *Surfaces of Higher Genus* – and *One-sided Surfaces*.

Franklin concluded with a review of the requirements of Heawood's *Empire* problem. He also mentioned the chromatic polynomial of Birkhoff and stated:

In a number of later papers he [Birkhoff] and Whitney have studied the properties of this polynomial, with a view to showing that it does not vanish for $p = 4$. However, while the polynomial is quite tractable for other values of p , and its values have even been interpreted for fractional values, it has not yet been shown that the polynomial cannot have 4 as a root.

In his conclusion, Franklin calculated that the probability of being unable to colour a map of over thirty-five countries with four colours is less than 1 in $10^{10,000}$, a strong argument in favour of the four-colour theorem, but still a long way from a proof. Of this paper, D C Lewis (see Chapter 8) wrote [8]:

In this purely expository paper is given a very comprehensive introduction to the four color problem and its generalizations. Practically all of the methods and results at present in the literature are touched upon to some extent. A few (particularly those pertaining to various classical results of Heawood) are given in considerable detail.

5.3 Henry Roy Brahana (1895–1972)

Another American mathematician who studied map colouring problems around this time was Henry Roy Brahana. He attended Dartmouth College in New Hampshire, and Princeton University, where he obtained his PhD in 1920 under the supervision of Oswald Veblen. After gaining his doctorate he went to the University of Illinois, becoming a professor and remaining there until his retirement.

His contributions to graph theory comprised three papers: *A proof of Petersen's theorem* [9], *The four-color problem* [10] and *Regular maps on an anchor ring* [11]. The first was written in 1917 during his postgraduate studies at Princeton. It provides a simpler proof of Petersen's 1891 theorem (see Chapter 3). It was published at a time when American scholars were beginning to take an interest in graph theory and colouring problems. The second paper, in 1923, presented the history of the four-colour problem from its origin in De Morgan's letter up to the early 1920s. This paper included some entirely human, rather than mathematical, observations such as:

The problem is still unsolved. It has afforded many mathematicians experience and very little else.

In this survey of the subject Brahana managed to mention most of the hitherto mathematicians who had worked on colouring problems, including De Morgan, Cayley, Kempe, Tait, Heawood, Wernicke, Petersen, Veblen, Birkhoff, Franklin and Errera.

The third paper was published in 1926 and studied the property that 'the group of a regular map of n k -sided regions is of order kn ' and:

... that any regular map on an anchor ring have triangular, quadrangular or hexagonal regions. In none of these cases is there a restriction on the number of regions. If the regions are triangular they appear six at a vertex; if quadrangular, four at a vertex; and if hexagonal, three at a vertex. Since two adjacent vertices are joined by a line and two adjacent regions are separated by a line there is a sort of duality between the maps of triangles and the maps of hexagons. The quadrangular maps are self-dual.

He therefore only dealt with hexagonal regions and quadrangular regions.

In part 1 Brahana assumed the existence of regular maps and used the theory of abstract groups to give conditions on n to state the existence of the map. He gave the following conditions for these two groups of maps:

A necessary condition that there exist a regular map of n hexagons is that there exist a group of order $6n$ generated by two operators of orders two and three respectively whose product is of order 6.

A necessary condition that there exist a regular map of n quadrangles is that there exist a group of order $4n$ generated by two operators of orders two and four respectively whose product is of order 4.

In the second part after stating:

We have shown that the existence of a regular map of n hexagons or quadrangles implies the existence of a group of order $6n$ or $4n$ of a certain type

(These groups being G_{6n} and G_{4n}). He showed that these conditions were sufficient to define the orders of regular maps. In the third part, after giving credit to G A Miller [12] for establishing the values of n for groups G_{6n} , Brahana developed the values of n for groups G_{4n} .

In the final part of the paper Brahana showed that there is a ‘uniqueness of the map for a given group’. He stated that:

As a result of our definition of regular map it follows that every such map has a group.

As a consequence of this statement then each group has a corresponding unique map.

5.4 Clarence Newton Reynolds, Jr. (1890–1954)

C. N. Reynolds undertook his postgraduate studies at Harvard, first supervised by Maxime Bôcher and then, on Bôcher’s death, gaining his PhD with a thesis entitled *On the zeros of solutions of linear differential equations* in 1919 under Birkhoff. After leaving Harvard he moved to the University of West Virginia, becoming head of the Mathematics Department in 1938 and remaining there until his retirement in 1946.

Although it has not been possible to ascertain how or when he learned of the four-colour problem, possibly through Birkhoff, he became another American who studied

colouring problems. Whilst at Harvard, and still possibly under the influence of Birkhoff, he published six graph theory papers between 1924 and 1932, five on colouring ([13], [14], [15], [16], [17]), and one concerned with circuits on polyhedra [18]. His note [12] was his first publication on graph theory and in it he offered a modification to the restrictions on the number of regions that can be coloured in four colours, given by Philip Franklin in his 1922 paper [3]

His papers [14], [15] and [16] are all concerned with map-colouring problems. In the first of these he considers, using analytical methods, the problem of developing the previously published geometric reductions of the map-colouring, including those by Kempe, Birkhoff and Franklin. He applies the theory of linear difference equations to examine the topological properties of irreducible connected configurations of pentagons. He shows that the reductions considered in the paper imply that a map on any spherical surface that is divided into not more than 27 regions may be coloured in four colours. He gave as examples two maps, each comprising 28 countries, both of which can be coloured in four colours but are also irreducible. This was to show that there was a correspondence between his analytical approach and the geometry considered. This result increased the Birkhoff number to 28.

Papers [15] and [16] gave a considerable amount of historical content of the four-colour problem and mention of those mathematicians who had made contributions to the quest for a solution, and particularly those who had contributed to the growing list of reducible configurations. The mathematicians mentioned included Euler, Kempe, Wernicke, Tait, Petersen, Birkhoff, Brahana, Errera, Frink, and particularly Franklin to whose work these two papers owe much. The intention of these papers is, as Reynolds states in the Introduction of [15]:

In this paper we shall develop some methods of so analysing the known geometric reductions of our problem as to discover and to prove some of their more important implications.

After stating that he would use reductions developed by Kempe, Birkhoff and Franklin, together with Errera's reduction of pairs of adjacent pentagons surrounded by hexagons, he continued:

Our fundamental method will be a systematic study of geometric operations which suffice to build any connected configuration of pentagons which exist in an irreducible map. Under these operations certain numerical topological characteristics are found to undergo well defined increments. Linear relations between these increments imply homogeneous linear difference equations which yield certain homogeneous relations between our topological characteristics.

Secondly we shall prove synthetically certain inequalities between the topological characteristics of an irreducible map, making use of the linear relations mentioned above to reduce them to more serviceable forms.

Applying these inequalities he proved the most important result contained in these papers:

Any map of a simply connected closed surface containing not more than twenty-seven regions may be colored in four colours.

5.5 Charles Edgar Winn

C E Winn was yet another American mathematician captivated by colouring problems who, between 1937 and 1940, published six papers on the subject. He spent most of his career at the Egyptian University in Cairo. In 1936, Winn along with Ismail Ratib, a colleague at the university, published a paper [19], presented to the International Congress of Mathematics in Oslo in 1936, which developed a generalisation of the Errera's reduction of the four-colour problem. In his two papers of 1937, one [20] returned to the four-colour problem, whilst the other [21] dealt with reducibility. In the latter paper he proved that a 5-vertex, all of whose neighbours have degree 6 is reducible, and a 5-vertex with one 5-neighbour, all the others being 6-vertices, is also reducible.

Continuing with reducible configurations in a 1938 paper [22], he proved that a 7-vertex with four consecutive 5-neighbours is reducible. In 1939, he published a paper [23] giving the history of the four-colour problem.

Perhaps his 1940 paper [24] was his most important contribution to graph theory. In this paper he proved that every map on a plane or sphere with 35 or fewer countries is colourable in four colours, thereby increasing the Birkhoff number to 36. To do this, he employed reductions by Birkhoff, Errera, Franklin, and himself, some of which he developed in the paper. As Franklin said in his review of the paper the work [25]:

... centres around a series of inequalities in which the number of pentagons in the map figure predominantly.

Conclusion

Whilst the mathematicians mentioned in this chapter made notable contributions to graph theory in the period between the two world wars, none of them came near the significance to the subject of Hassler Whitney. The next chapter is devoted to his work in graph theory.

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Chapter 6

Hassler Whitney

Hassler Whitney was probably the leading American mathematician of the second quarter of the twentieth century; his insightful ideas had considerable influence in many areas of mathematics; indeed, he founded and provided the tools for a number of mathematical areas, particularly algebraic topology. He came from a long line of distinguished men and women, some were privileged, some learned, many highly intelligent, and most of whom left tangible legacies of knowledge and achievement (see his biography in Appendix I). Whitney made significant contributions to graph theory beginning with his doctorate on graph theory and the dozen papers on the subject he presented to the American Mathematical Society in the years 1930 to 1934. During the last twenty years of his life Whitney devoted considerable time and effort to improving the teaching of mathematics, particularly at the elementary school level.

6.1 Hassler Whitney (1907–1989)

Hassler Whitney was born on 23 March 1907 in New York City into an extremely influential family. He was only 3 years old at the time of his father's death on 5 January 1911, and only 2 when Simon Newcomb (his maternal grandfather) died, so their direct influence was limited. He gained bachelor degrees in physics in 1925 and music in 1929, both from Yale University.

After graduating in physics, he visited Göttingen University; Germany was then considered as the foremost centre for mathematics in the world. During his time there he decided to change from physics to mathematics. In his response to the Steele Prize in 1985, he said 'in physics it seems you have to remember facts, so I gave it up and moved into

mathematics'. He became very keen on logic, and developed an interest in graph theory and the four-colour problem whilst in Germany [1].

Following his graduation in music, Whitney moved to Harvard University to undertake his doctoral studies. Most authorities name G D Birkhoff as being his supervisor during his postgraduate study at Harvard, but in 1984, during an interview [2] conducted by Albert Tucker and William Aspray, Whitney remembered it otherwise:

Tucker: Was it [James W] Alexander that supposedly was your supervisor or [Solomon] Lefschetz or [Oswald] Veblen?

Whitney: It was essentially Alexander. I don't remember if there was a formal requirement that I have a supervisor, but he served as one most of the time.

Two of the most credible authorities on the subject are the American Mathematical Society and Birkhoff's son Garrett. The AMS publication *The Presidents* [3] includes the statement:

The following 38 students have prepared their thesis for the doctorate in association with Professor Birkhoff at Princeton, Yale and Harvard U., and Radcliffe C., 1911–1937.

The list included Hassler Whitney. Garrett Birkhoff's article *Mathematics at Harvard, 1836-1944* [4] stated:

During the next two decades [from 1912] G D Birkhoff would supervise the PhD theses of a remarkable series of graduate students.

He then gave a partial listing that included Hassler Whitney. Whoever his supervisor was, Whitney wrote up Birkhoff's 1930 paper on chromatic polynomials for him.

It is interesting to note that, for a second time during the interview, Whitney failed to recall Birkhoff, especially as the time being discussed was when Whitney was carrying out research under him. Another part of the interview was:

Aspray: Which of the faculty members did you work with most closely?

Whitney: I wouldn't say I worked with anybody really. I saw a lot of various people and chatted with them, but never really talked mathematics that much. ...

Tucker: And indeed you may very well have attended the dedication ceremony, because it was held there in October 1931. ... And for that occasion some visiting people came: G D Birkhoff was there, G A Bliss was there, and there was a one-day symposium of mathematical talks.

Whitney: That has entirely gone out of my mind.

It may be that in old age (he was 77 at the time of the interview) his memory had somewhat deteriorated. However, as he provided his interviewers with considerable detail from his past, the reason for his lapses of memory regarding his supervisor was perhaps a more personal one. Harvard records that Birkhoff was Whitney's supervisor.

Another indication of the closeness between Whitney and Birkhoff around this time is that Whitney's doctoral thesis, *The Coloring of Graphs*, for which he was awarded his PhD in 1932, was inspired by Birkhoff's 1912 paper on chromatic polynomials, as was his work on the logical expansion [6]. Part of his thesis was published in 1932 as a paper [11] of which it was said in [1]

In this paper he continued his study of chromatic polynomials, showing that the numbers m_j (in the notation of [9D]) may be obtained by considering only the non-separable subgraphs, instead of the much larger set of all subgraphs. — (9D was the reference to [6]).

Overlapping these studies was a fellowship from the National Research Council, which allowed him to go to Princeton University (1931–1933). Several of his subsequent graph-theoretical papers were extensions of his postgraduate work and many were written during his time at Princeton.

In 1930, Whitney had become an instructor in mathematics at Harvard and returned to that position in 1933, was appointed an assistant professor in 1935, an associate professor in 1940 and a full professor in 1946, a position he held until 1952.

As mentioned earlier, Whitney made unique contributions to many areas of mathematical learning. His work on graph theory will be commented on later in this chapter, but his other studies included combinatorics, characteristic classes, classifying spaces, stratifications, manifolds, cohomology, fibre bundles, differential topology,

matroids, and geometric integration; in each of these subjects, he advanced knowledge and left results for others to build on. Many of these areas of study owe their parentage to Whitney and his work set the standards for others entering these fields during the second half of the twentieth century. This is certainly true of algebraic topology, as he is regarded as one of the recent founders of this subject.

In manifolds his work was seminal. His paper on this topic was influenced by previous work by both Oswald Veblen and J H C Whitehead (1904–1960) a British mathematician who was highly regarded for his work in the study of knots, and was a considerable step forward in the understanding of manifolds. Whitney was an invited speaker at a Topology Congress in Moscow in 1935, where he gave two talks.

Whitney later claimed that one of his more trivial papers, (*On the abstract properties of linear independence*), became a basis of a ‘large new branch of combinatorial theory’ of which he said in 1985: ‘I do not attempt to understand’. Perhaps a little too self-deprecating a remark? Like many other mathematicians, Whitney contributed to the war effort in World War II. He served as part of the Applied Mathematics Group at Columbia University and was primarily responsible for that part of the programme studying the use of rockets in air warfare. His duties included the integration of the work being carried out by the groups at Columbia and Northwestern Universities in the general field of fire control for airborne rockets. He also carried out liaison work with the Fire Control Division of the National Defense Research Committee in rocketry and with numerous Army and Navy units — notably, the Naval Ordnance Test Station, the Dover Army Air Base, the Wright Field Armament Laboratory, the Naval Bureau of Ordnance, and the British Air Commission.

In 1952 he moved to the Institute for Advanced Study, Princeton, as Professor of Mathematics, a position he held until his retirement in 1977 when he was made Professor Emeritus, holding this position until his death.

It was said of Whitney that he was friendly and shy, but with a wry sense of humour, and although a little reticent, was straightforward and displayed artlessness. He was a man who very noticeably did not get upset easily when things went wrong. This was commented upon by those who witnessed such incidents as when he dropped his spectacles in a lake, locked the car keys inside his car, or had his wallet stolen by a pickpocket on the New York subway. He was tolerant and patient: he did not teach or preach, but rather provided suggested avenues of thought and gave opportunities for reflection. He was a man who listened to what others had to say. He certainly possessed considerable capacity for deep and complex thought whilst perusing mathematical ideas.

In graph theory, Whitney invented numerous tools that would be of lasting importance to graph theory in general, and also to the eventual solution of the four-colour conjecture in particular. These included his theory of duality, used to characterise planar graphs, and his theory of linear dependence and independence introducing the theory of matroids.

6.2 Whitney's graph theory papers

In the early 1930s he wrote some dozen papers on topics within graph theory; these papers are reviewed in the following pages in the order of their presentation to the American Mathematical Society, the first in February 1930 and the last in September 1934. Some of these, such as the third, ninth, and twelfth, are combinatorial in content and deal with planarity, whereas others, including the first, second, fourth and eleventh, are related to colouring problems. These papers are, with their year of publication:

- 1 *A theorem on graphs.* (1931)
- 2 *A logical expansion in mathematics.* (1932)
- 3 *Non-separable and planar graphs.* (1932)
- 4 *The coloring of graphs.* (1932)
- 5 *Congruent graphs and the connectivity of graphs.* (1932)
- 6 *A characterization of the closed 2-cell.* (1933)
- 7 *A set of topological invariants for graphs.* (1933)

- 8 *On the classification of graphs.* (1933)
- 9 *Planar graphs.* (1933)
- 10 *2-isomorphic graphs.* (1933)
- 11 *A numerical equivalence of the four color problem.* (1937)
- 12 *On the abstract properties of linear independence.* (1935)

Paper 1 *A theorem on graphs*

Whitney's first graph-theoretical paper [5] states and proves that every triangulation of a graph has a *Hamiltonian cycle*, and introduces the concept of a *dual graph*. It was presented to the American Mathematical Society on 22 February 1930 and had a notable significant result, succinctly put in the book *Graph Theory 1736–1936* [1] as:

He (Whitney) proved that if we have a cubic plane map with no loops, then its dual graph must be Hamiltonian. So, in the dual form of the four-colour conjecture — the assertion that every planar graph may be coloured with four colours — it is sufficient to consider only Hamiltonian planar graphs. This is undoubtedly a deep result, as anyone who tries to understand Whitney's paper will agree, and its implications have yet to be exploited fully.

The paper was in four parts. The first, *Results of this paper*, stated his two main theorems that were proved in the second and third parts. The final part, *Further remarks* dealt with the necessity of a particular assumption in the statement of Theorem II.

In the first part of the paper, Whitney defined his graphs with vertices as points on a plane or sphere and edges as arcs. The surface was thus divided into connected regions with elementary polygon circuit boundaries. He then applied a *triangulation* by dividing his graph into circuits of elementary triangles.

After setting out his definitions, Whitney stated his main theorem:

Theorem I. Given a planar graph composed of elementary triangles, in which there are no circuits of 1, 2, or 3 edges other than these elementary triangles, there exists a circuit which passes through every vertex of the graph.

(This circuit is a Hamiltonian cycle, although Whitney did not call it that.) The first part concluded with a statement of the four-colour problem.

In the second part, Whitney proved Theorem I. In the following quotations from the paper 'line' means 'path' and 'touch' means 'adjacent'. He first stated a lemma:

Lemma: Consider a circuit R in a graph of the type considered in Theorem I, together with the vertices and the edges on one side, which we shall call the inside. Let A and B be two distinct vertices of R , dividing R into the two parts R_1 and R_2 , in each of which we include both A and B . Suppose

- (1) No pair of vertices of R_1 touch each other inside R (are joined by an edge which lies inside R), and
- (2) Either no pair of vertices of R_2 touch each other inside R , or else there is a vertex C in R_2 distinct from A and B , dividing R_2 into the two parts R_3 and R_4 , in each of which we include C , such that no pair of vertices of R_3 and no pair of vertices of R_4 touch each other inside R .

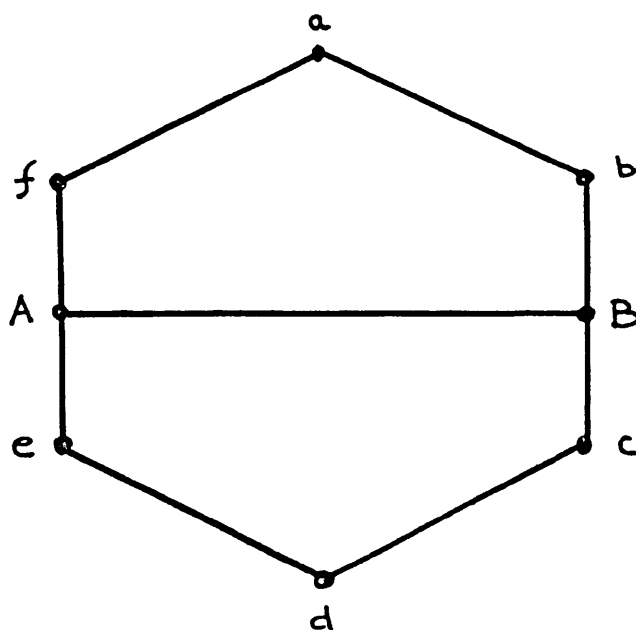
Then we can draw a line from A to B , passing only along edges of and inside R , and passing through each vertex of and inside R once and only once.

He took as his basis the circuit R from the graph defined at the beginning of the paper. This circuit and the additional circuits defined in the lemma are:

circuit R is $a, b, B, c, d, e, A, f, a$

circuit R_1 is A, f, a, b, B, A

circuit R_2 is A, e, d, c, B, A



Whitney went on to say:

In brief, if we can divide the circuit R into either two or three parts, such that in any part, including end vertices, no pair vertices touch each other inside R , we can draw the required curve from any one end vertex to any other end vertex of these parts.

He also said that with these conditions, there is a Hamiltonian path from A to B in R_1 . Similarly there is a Hamiltonian path from B to A in R_2 . Combining these Hamiltonian paths gives a Hamiltonian cycle. The lemma was proved by assuming that it was true for:

... all circuits which, with the vertices inside, contain m vertices, $m = 3, 4, \dots, n-1$. It is obvious for the case where $m = 3$. We will prove it for all circuits which, with the vertices inside, contain n vertices. Then by mathematical induction, it is true in general.

The proof of the lemma was divided into four parts ‘according to which pairs of vertices of the circuit touched inside the circuit’. The proof of the lemma included a complicated argument using four defined cases, and Theorem I followed from the lemma.

Whitney went on to state ‘a theorem on maps’, his Theorem II, which he asserted was ‘deducible immediately’ from Theorem I:

Theorem II. Given a map on the surface of a sphere containing at least three regions in which:

(A₁) The boundary of each region is a single closed curve without multiple point,

(B) Exactly three boundary lines meet at each vertex,

(A₂) No pair of regions taken together with any boundary lines separating them form a multiply connected region,

(A₃) No three regions taken together with any boundary lines separating them form a multiply connected region,

we may draw a closed curve which passes through each region of the map once and only once, and touches no vertex.

In the third part Whitney set out his ‘Proofs of the theorems on maps’, starting by showing that any graph of the type being considered has a *dual graph*, with the following properties (α), (β) and (γ):

- (α) Each vertex touches at least three other vertices in cyclic order, distinct from each other and distinct from the first,
- (β) If a touches b and next c , then b touches c and next a ,
- (γ) There are no triangles other than elementary triangles.

Whitney coupled Theorem II with a lemma, which he also proved in Part 2 of the paper, to address what he called a ‘conundrum’. This conundrum was ‘Suppose a man, living in a certain country (state), wishes to visit all the countries about him, but does not wish to pass through any country more than once on his voyage’ — that is, a closed cycle. This is similar to the Icosian Game devised by Hamilton.

He then applied Theorem I to the dual graph, which gave a closed curve of vertices passing through every region of the map. He continued by considering maps of the type defined in Theorem II, and concluded, as observed by Kempe in 1879, that if the dual graph can be coloured using four colours, then the corresponding map can also be coloured with four colours. These comments on the four-colour problem formed a prelude to the contents of the second of the papers in this series.

Paper 2 *A logical expansion in mathematics*

In the second paper [6] being considered, the part of which concerned graph-theory, Whitney developed the ancient *principle of inclusion and exclusion* of de Moivre and de Montmort (see Appendix 1). He used a quantitative approach to the four-colour problem and was influenced by, and developed, work of G D Birkhoff. But whereas Birkhoff’s work had applied to maps, Whitney concentrated on graphs. Additionally, the paper showed how the principle of ‘logical expansion’ might be applied to a number of other topics in mathematics. This paper was presented to the American Mathematical Society on 25 October 1930.

In the *Introduction* Whitney defined his basic logical notation, which he used not only in this paper but, as he noted, in his yet-to-be published paper *Characteristic functions and*

the algebra of logic [7] (This paper, not reviewed here, was published by the American Mathematical Society in 1933 and referred to by Whitney as CF). Whitney took a finite set of objects, his example was ‘books on a table’, comprising those that had or had not a property A , his example ‘say of being red’. If n is the total number of objects in the set, $n(A)$ is the number with property A , and $n(A')$ is the number of those not having property A , then, clearly, $n(A') = n - n(A)$. That is: given a set and a number of properties, how many elements of the set have none of these properties.

The principle of inclusion and exclusion, also known as the sieve principle, provides a method to obtain the number of elements in the union of a given group of sets, the size of each set and the size of all possible intersections among the sets. The general formula for two sets is:

$$|\bigcup_{i=1}^2 A_i| = |A_1| + |A_2| - |A_1 \cap A_2|$$

and for 3 sets is:

$$|\bigcup_{i=1}^3 A_i| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Whitney was presenting an alternative way of writing the inclusion-exclusion principle so that for a set of objects n that has properties A and B where $n(AB)$ denote the number of objects with both properties A and B then the number of objects with neither property is:

$$n(A' B') = n - n(A) - n(B) + n(AB)$$

and for a set of objects n that has properties A , B and C the formula for the number of objects with none of these properties is:

$$n(A' B' C') = n - [n(A) + n(B) + n(C)] + [n(AB) + n(AC) + n(BC)] - [n(ABC)]$$

Whitney commented that the extension of these formulas to the general case where any number of properties was considered is simple. He added that the general case was well known to logicians and should be better known to mathematicians. (That may well have

been the case in 1930, but surely not now in the early years of the twenty-first century.)

In the second part of this paper, he presented an inductive proof of the general formula for the number of objects without certain properties in terms of the numbers of objects with several of these properties.

In part three, *The measure of characteristic functions*, Whitney related this paper to his paper *Characteristic functions and the algebra of logic* mentioned previously. He developed a class of formulas which contains the logical expansion as a special case, making use of characteristic functions, saying:

With each element x of a set of n objects R we associate an integer $A(x)$, either positive, negative or zero. We define the measure of A by the equation

$$n(A) = \sum_{x \text{ in } R} A(x).$$

Suppose A were one for certain elements of R (which elements form a set A') and zero for the rest. Then A is the characteristic function of A' , and $n(A)$ is just the number of elements in A' .

In part four, *On prime numbers*, Whitney used the logical expansion formula to derive an expression for the number of integers less than or equal to a given number and not divisible by any of a given set of primes.

In part five, *A problem in probabilities*, Whitney addressed the probability that no card of a second pack of cards to be dealt out on top of a first pack laid out in a row will lie on the same card as the first row. The probability is the sum of the first $m + 1$ terms in the series for $1/e$ where m is the number of cards in the pack. This is the derangement problem, studied earlier by de Moivre and Euler (see Appendix 1).

In the sixth part, *On the number of ways of coloring a graph*, he used the logical expansion formula to consider a set of objects and pairs of these objects. He then supposed that there were a fixed number of colours λ available and that these colours be assigned to the vertices such that any two vertices which are joined by an arc be of different colour. This he called an admissible colouring of the graph and stated that:

We wish to find the number $M(\lambda)$ of admissible colourings, using λ or fewer colors.

He then considered a special case of the general problem — the four-colour problem. He developed an expression for the number $M(\lambda)$ of ways of colouring a map on a sphere with a fixed number of λ of colours. He stated that the ‘graph (which is just a dual graph of the map) is constructed by placing a vertex in each region of the map, and joining two vertices by an arc if the corresponding regions have a common boundary’. By building on the work of Birkhoff, Whitney developed the function M , a polynomial function of λ , which included the terms m_{ij} , which were defined as the number of subgraphs of rank i , nullity j . The formula is:

$$M(\lambda) = \sum_{i,j} (-1)^{i+j} m_{ij} \lambda^{V-i},$$

where V is the number of vertices of the graph, s the number of arcs and p the number of connected components of the graph; i is its *rank*, defined as $i = V - p$, and j is its *nullity* defined as $j = s - i = s - V + p$.

Whitney had developed a simpler method for determining the coefficients of the chromatic polynomial than Birkhoff’s in [8]. Additionally, although implicit in Birkhoff’s paper, it was Whitney who specifically drew attention to the fact that the coefficients of the polynomial alternate in sign.

In part 7, *The m_i in terms of the broken circuits of G* , Whitney showed that the coefficients m_i of $M(\lambda)$, can be found directly in terms of the ‘broken circuits’ and subgraphs of a graph and that m_i are related to m_{ij} by:.

$$m_i = \sum_j (-1)^{i+j} m_{ij}$$

A *broken circuit* is a cycle with one edge removed. As an example, he considered a graph G with vertices a, b, c, d and defined the arcs ab, ac, bc, bd, cd , stipulating that they be in that order. He listed the cycles in G , in each case listing their edges in the order fixed previously. From these cycles he found broken circuits P_i by deleting the last arc of each circuit. From these he stated:

... the number $(-1)^i m_i$ is the number of subgraphs of G of i arcs which do not contain all the arcs of

any broken circuit.

He developed his argument arriving at the formula:

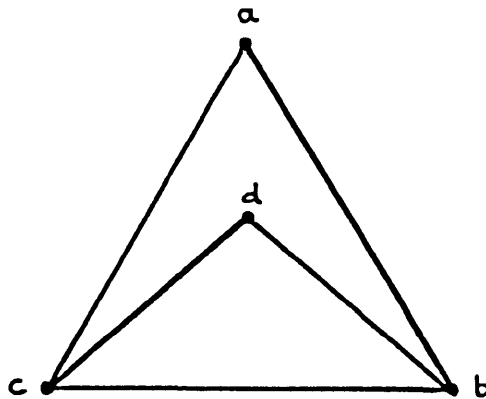
$$M(\lambda) = \sum_i (-1)^i l_i \lambda^{V-i},$$

where V is the number of vertices, and l_i is the number of subgraphs with $l_i = (-1)^i m_i$.

Whitney took as an example the complete graph K_3 on vertices a, b, c , with arcs ab, ac, bc , and one broken circuit: ab, ac . This resulted in one subgraph with no arcs ($m_0 = 1$), three subgraphs with a single arc ($m_1 = 3$), three subgraphs with 2 arcs, one of which contains the broken circuit ($m_2 = 2$), and one subgraph of 3 arcs containing the broken circuit. With $V = 3$:

$$M(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2).$$

At the beginning of part 7 Whitney also gave as an example the following graph G :



and developed the equation:

$$M(\lambda) = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda = \lambda(\lambda - 1)(\lambda - 2)^2$$

This was an important paper for graph theory, which advanced the quantitative approach to the four-colour problem, and included his explicit formula for the values of $P(\lambda)$.

Paper 3 Non-separable and planar graphs

This long and complex paper [9] on planarity was in two parts. The first part, which contained the main result of the paper, concerned non-separable graphs, detailed nineteen

theorems, and a lemma dealing with separability of graphs. The second part, developed from Euler's polyhedra formula, related to duals and planar graphs and included thirteen theorems and three lemmas.

Whitney presented his third paper to the American Mathematical Society on 25 October 1930, together with his previous paper [6]. A three-page summary of the main results of this paper, without proofs, was published by the National Academy of Sciences in 1931 [10].

In both parts of the paper he utilised the rank, r , R , and nullity, n , N , of graphs, together with their associated equations, as aids to many of the formulas. Whitney's approach was purely combinatorial and concerned abstract graphs only.

In the first part of the paper he began by setting out the definitions, including those for rank and nullity, which he considered fundamental to his work. He then developed the decomposition of separable graphs, cycles in graphs and the construction of non-separable graphs. The important theorems in this section, which Whitney used to prove the main result of the paper, are:

If G is not non-separable, we say G is *separable*. Thus, a graph that is not connected is separable.

Suppose some connected piece G_1 of G is separable. If H_1 and H_2 joined at vertex a form G_1 , we say a is a *cut vertex* of G .

Theorem 5. A necessary and sufficient condition that a connected graph be non-separable is that it have no cut vertex.

Theorem 6. Let G be a connected graph containing no 1-circuit. A necessary and sufficient condition that the vertex a be a cut vertex of G is that there exist two vertices b, c in G , each distinct from a , such that every chain from b to c passes through a .

Theorem 8. A non-separable graph G containing at least two arcs contains no 1-circuit and is of nullity > 0 . Each vertex is on at least two arcs.

Theorem 9. Let G be a graph nullity 1 containing no isolated vertices, such that the removal of any arc reduces the nullity to 0. Then G is a circuit.

Theorem 18. If G is a non-separable graph of nullity $N > 1$, we can remove an arc or suspended chain from G' , leaving a non-separable graph g of nullity $N - 1$.

Whitney used the term non-separable and as explained in [1]:

Unfortunately, he used this term in a non-standard way. We say that a graph is **non-separable** if it is connected and cannot be disconnected by removing a single vertex (a **cut vertex**). In Whitney's sense, a 'component' is a maximal non-separable part of a graph. Nowadays we call this a **block**.

In the second part, Whitney characterised planar graphs as those with a dual. This dual was not the usual geometrical dual, but one defined in terms of cycles and cutsets. He said that 'We used formerly the word homeomorphic' and stated that 'Throughout this section R, R', r, r' , etc., will stand for the ranks of G, G', H, H' , etc. respectively, with similar definitions for V, E, P, N . (V = number of vertices, E = number of edges, P = number of connected pieces, N = nullity).

Whitney pointed out that:

Up to now, we have been considering abstract graphs alone. However, the definition of a planar graph is topological in character. This section may be considered as an application of the theory of abstract graphs to the theory of topological graphs'.

His *Definitions* included:

A topological graph is called *planar* if it can be mapped in a (1, 1) continuous manner on a sphere (or a plane)

and the following theorem was included in this section:

Theorem 29. A necessary and sufficient condition that a graph be planar is that it have a *dual*.

Whitney developed relationships between a map formed by a planar graph G drawn in the plane and a dual map, formed by the geometrical dual G^* . Also if G was non-separable then $G^{**} = G$; this result does not hold if G has more than one 'component'. From this, it can be stated that a graph is planar if and only if it has what is now known as a *Whitney dual*. A Whitney dual is a combinatorial definition of abstract duality which concerns the cycles and cutsets of two graphs and satisfies the requirements of a dual graph that is planar. Because of this result, Whitney said that for the rest of the paper, the word 'graphs' could apply 'equally well to either abstract or topological graphs'.

Whitney concluded the paper with a proof of Euler's formula which he derived from rank and nullity, similar to the method used by Veblen (see Chapter 4). Whitney stated:

Map any connected planar graph G on a sphere and constructed its connected dual G' as described in the proof of Theorem 29.

With F being the number of regions (or faces) and E the number of arcs of the planar graph G and using the rank R and nullity N of both graphs he developed the formulas:

$$R' = N, R = V - 1, R' = V' - 1, V' = F.$$

Hence $V - E + F = R + 1 - E + N + 1 = 2$, which is Euler's formula.

Paper 4 *The coloring of graphs*

This paper [11], which dealt with chromatic polynomials, was a revised version of Whitney's doctoral thesis and was presented to the American Mathematical Society on 25 October 1930, the same day that he presented his two previous papers. Like the previous paper, a summary of the main results of this paper was also published by the National Academy of Sciences in 1931 [12].

This new paper, in which Whitney again worked on chromatic polynomials, comprised an introduction and five parts. It studied the coefficients m_{ij} that appear in the chromatic polynomial formulas for $M(\lambda)$, and gave a method of calculating them. It was in this paper that he demonstrated the important concept that in order to find the numbers m_{ij} , where i is the rank and j the nullity, it was necessary to consider only non-separable subgraphs, rather than the set of all subgraphs.

The *Introduction* began by referring to the formulas for $M(\lambda)$, the number of ways of colouring a graph with λ colours, contained in his logical expansion paper [6]. It concluded by referring to the four-colour problem and stating the theorem:

If G is a planar graph and G' is a dual of G , and m_{ij} and m'_{ij} are their numbers, then $m'_{ij} = m_{R-j, N-i}$

— this followed from the definition of dual graphs in his paper on planar graphs [9]. He then considered:

... the class of graph G with numbers m_{ij} for which the above numbers m'_{ij} are also the numbers for the graph G' ; this class then includes all planar graphs.

and continued with the proposal that implied the four-colour theorem for the class of graphs G :

For any graph G of the above class, $\sum_{i,j} (-1)^{i+j} m_{ij} 4^{V-i} > 0 \quad m_{01} \neq 0.$

Whitney stated:

This proposition is stronger than the four color map theorem; for there are graphs in the above class which are not planar.

In the first part of this paper, *The polynomial $M(\lambda)$* , Whitney made use of his work in [6] and paper [7] which he referred to as CF. He explored the polynomial $M(\lambda)$ in terms of the number m_{ij} and calculated the sum of all the coefficients for a particular subgraph G which contains i arcs (i being the rank of G).

In the second part, *m_{ij} in terms of non-separable subgraphs*, Whitney showed that when calculating m_{ij} it is not necessary to count all the subgraphs of a graph, but simply to count the non-separable subgraphs. Whitney defined a non-separable subgraph as:

We say a subgraph is non-separable if it becomes so when any isolated vertices there may be are dropped out. This operation alters neither the rank nor the nullity of the subgraph.

In the third part, *The transformation T* , Whitney addresses the theory of algebraic transformations and details two versions of the transformations, the first to calculate for a single subscript i (rank) and the second for two subscripts ij (rank and nullity). The transformations are:

$$m_i = \sum_k 1/k! R_i^k(f) \text{ and } m_{ij} = \sum_k 1/k! R_{ij}^k(f),$$

where $R_i^k(f)$ is the sum of all terms formed by multiplying together k numbers $f_{j_1}, f_{j_2}, \dots, f_{j_k}$ whose subscripts (which need not be distinct) add up to i , and R_{ij}^k is defined as $R_i^k(f)$ is defined, with the extra condition that the first subscripts add up to i , and the second ones add up to j .

In the fourth part he allows the elements that were used to develop m_i in the previous part to be graphs, and again sets out the procedure for calculating m_{ij} .

In part five, *Calculation of the f_{ij} and m_{ij}* , Whitney states that N_{ij} is the number of non-separable subgraphs of rank i , nullity j and includes diagrams of all the non-separable graphs with ranks 1 to 4. He also details the calculations of values f_{ij} and m_{ij} for some of these graphs.

The paper concludes with a short discussion on regular maps, in relation to the four-colour problem, where he defines regular maps as:

- (1) All the vertices are triple.
- (2) There are no rings of one, two, three or four regions; that is, no set of four or fewer regions, together with all boundary lines separating them, form a multiply connected region.
- (3) There are no regions of five regions except about a single region.

He developed many specific values for f_{ij} and m_{ij} and ended with a calculation of the polynomial $M(\lambda)$ for the dodecahedron graph, stating:

The regular map of the fewest number of regions is the dodecahedron, containing twelve regions, each touching five others. For this map

$$V = 12, E = 12, T = 20, Q_5 = 12, R_6 = 40.$$

We can thus calculate the first six coefficients. The whole polynomial, which was calculated by direct means, is

$$M(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)[\lambda^8 - 24\lambda^7 + 260\lambda^6 - 1670\lambda^5 + 6999\lambda^4 - 19698\lambda^3 + 36408\lambda^2 - 40240\lambda + 20170].$$

Here V is the number of regions in the map (number of vertices in the dual graph); $E = 3(V - 2)$, the number of boundary lines (number of arcs); $T = 2(V - 2)$, the number of vertices ($=N_{21}$); Q_5 is the number of 5-sided regions (number of vertices of degree five) and R_6 is the number of rings of six regions, no two having a common boundary unless they are adjacent in the ring (number of 6-circuits in which only vertices adjacent in the circuit are joined by arcs in the graph):

This was a lengthy and significant paper which advanced the algebraic approach to the four-colour problem.

Paper 5 *Congruent graphs and the connectivity of graphs*

The opening paragraph of this paper [13], presented to the American Mathematical Society, on 28 February 1931, gives a succinct description of its contents:

We give here conditions that two graphs be congruent and some theorems on the connectivity of graphs, and we conclude with some applications of dual graphs.

For the definition of congruent graphs Whitney made reference to a previous paper on planar graphs [9] which included:

Congruent graphs. We introduce the following

Definitions. Given two graphs G and G' , if we can rename the vertices and arcs of one, giving distinct vertices and distinct arcs different names, so that it becomes identical with the other, we say the two graphs are congruent.

He stated that ‘The definitions and results’ of his paper [9] ‘will be made use of constantly’ and Whitney used the formulas contained in that paper to develop the proofs in this paper.

In the first section, *Congruent graphs*, Whitney considered graphs with no 1-cycles (loops) or 2-cycles (multiple arcs) i.e. simple graphs, and stated the theorem:

Theorem 1. Let G and G' be two connected graphs, neither of which consists of three arcs of the form ab, ac, ad . Let there be a 1–1 correspondence between their arcs so that to any two arcs having a common vertex in one graph correspond two arcs having a common vertex in the other. Then G and G' are congruent.

His proof examined exhaustively the different configurations of the defined graph, dealing with both non-separable and separable graphs.

He then looked at the congruency of two triply-connected graphs, again with a 1–1 correspondence between their arcs, but this time the correspondence was between sets of arcs forming a cycle, and stated the theorem:

Theorem 2. Let G and G' be two triply connected graphs, and let there be a 1–1 correspondence between their arcs so that to any set of arcs forming a circuit in one graph, corresponds a set of arcs forming a circuit in the other. Then G and G' are congruent.

At the conclusion of the proof of Theorem 2 he pointed out that the theorem does not hold for all non-separable graphs, and referred to an example given in his paper on logical expansion [6]. Theorem 3 also dealt with the congruency of two triply-connected graphs with a 1–1 correspondence between their arcs, but this time it was between arcs that form a subgraph of nullity 1. Whitney used Theorem 9 of [6] to prove that circuits correspond to circuits and therefore the above-mentioned Theorem 2 applies.

In the second section, *The connectivity of graphs*, Whitney let his graphs contain 2-cycles, but not 1-cycles. In this section, Theorem 4 gives conditions for a graph to be n -tuply connected, while Theorem 5 characterised graphs that are not so connected. Theorem 6 gives the conditions for a n -tuply connected graph to be $(n - 1)$ -tuply connected, by deleting a vertex.

In the last section of the paper, *Applications to dual graphs*, Whitney proved the requirements for two graphs to be dual graphs in terms of their connectivity, and in Theorem 9 he explored their connectivity. Theorem 10 proved that a dual G' of a triply-connected graph G containing no 1-cycles or 2-cycles, has similar properties, and Theorem 11 went on to prove that a triply-connected planar graph, containing no 1-cycles or 2-cycles, has a unique dual.

Paper 6 *A characterization of the closed 2-cell*

This paper [14] is concerned with the classification of the 2-cell, a set homeomorphic to a closed disc. It was presented to the American Mathematical Society on 31 October 1931, although it was not published in their *Transactions* until 1933. Because this paper is combinatorial in content it has been included in this thesis.

The fundamental theorem of the paper is his Theorem I. As Whitney stated ‘The fundamental theorem is partly of a combinatorial and partly of a continuity nature’:

Theorem I. Let R be a continuous curve containing the simple closed curve J , such that

(1) J is irreducibly homologous to zero in R , and

(2) If γ is an arc with just its two end points a and b on J , then $R-\gamma$ is not connected.

Let R' and J' be defined similarly. Then R and R' are homeomorphic, with J corresponding with J' .

That R is a closed 2-cell then follows immediately from the following theorem. We note that J corresponds with the circle, that is, J is the boundary of R .

Whitney provided an extremely detailed proof of this theorem I, which included the posing and solving of nineteen lemmas. He also provided a shorter proof of his Theorem II, which utilises the Jordan Theorem, and read:

Theorem II. If I is a circle in the plane and S is I with its interior, then S and I satisfy the conditions prescribed for R and J in the above theorem. [Theorem I]

Section 4 included, as Whitney said in the *Introduction*, ‘The exact meaning of Condition (1) of Theorem I is given in 4’:

Suppose the closed set R contains the simple closed curve J . If for every $\varepsilon > 0$ there is a $\delta > 0$ such that any $(\delta, 1)$ -cycle on J is ε -0 in R , then we say that $J \sim 0$ in R . If J is ~ 0 in R but is not ~ 0 in any proper closed subset of R containing J , then we say that J is *irreducibly* ~ 0 in R .

Paper 7 *A set of topological invariants for graphs*

This paper [15] dealt with topological graphs and gave ‘a set of topological invariants which come from a set of m_{ij} defined by the author’ from [6]. The paper, presented by Whitney to the American Mathematical Society on 28 December 1931, was brief and made numerous references to four of the previously considered papers [6, 9, 11, 13].

In his *Introduction* he defined a topological graph and, by considering arcs and vertices as abstract elements, he developed an abstract graph. A topological graph as defined by Whitney, ‘is a point set consisting of a finite number of points, or vertices, and a finite number of open arcs (topological images of an open segment) which do not

intersect, joining pairs of these points'. He went on to say 'If we consider the vertices and arcs as abstract elements instead of as point sets, and name the two vertices which each arc joins, we obtain the corresponding abstract graph G '.

The paper had four further parts. The first, *The invariants*, Whitney introduced as:

Given the table of the sum m_{ij} for a graph G , if we sum over elements in each row with alternating signs, we get the m_i , the coefficients of the polynomial $M(\lambda)$ for the number of ways of coloring G in λ colors. Suppose, instead, we sum over the columns; we get a set of numbers p_i which we shall show are topological invariants of the graph. The numbers are, if G is of rank R and nullity N :

$$p_i = \sum_j (-1)^{i+j} m_{R-j, N-i}.$$

The number of non-zero numbers p_i (if there are any) equals one plus the nullity N of G . For a graph with no arcs, $p_0 = 1$, and $p_i = 0$, $i \neq 0$.

The following section, *Broken cut sets of arcs*, contained a proof that $(-1)^i p_i$ is the number of subgraphs of i arcs of G which do not contain all the arcs of any broken cut set of arcs'.

The next part, *Separable graphs*, demonstrated how to calculate the numbers p_i for a graph G that is the union of two graphs, G' and G'' with at most a single common vertex. In the last section, *Completeness of the invariants*, Whitney proved that if two graphs are 2-homeomorphic, then the corresponding values of p_i are equal. In this paper, after the words '2-homeomorphic' Whitney wrote '(see the following paper)', which is paper [16] and the comments on this paper that follow will include Whitney's definitions of isomorphic and homeomorphic graphs.

Paper 8 *On the classification of graphs*

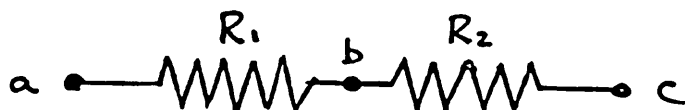
This paper built on the work of Ronald M Foster, an employee of the Bell Telephone Company, in his 1932 paper on the graph theory underlying electrical networks [17]. It was presented to the *American Mathematical Society* under the title *Basic graphs* on 28 December 1931, although when published it was entitled *On the classification of graphs* [16].

Whitney's opening paragraph succinctly outlined the contents of the paper:

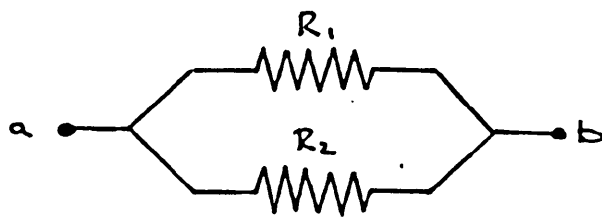
Introduction. R M Foster has given an enumeration of graphs, for use in electrical theory. He uses two distinct methods, classifying the graphs according to their nullity, and according to their rank. In either case, only a certain class of graphs is listed; the remaining graphs are easily constructed from these. In the present paper we give theorems sufficient to put the first method of classification on a firm foundation.

Whitney went on to say that the 'manner of constructing the graphs, and in particular, the important notation of basic graphs, is due to Foster', but 'the definition of elementary graphs and the proofs are, in general, due to the author' [Whitney].

Whitney stated that the terminology of electrical circuit theory would be used: ab , bc are in *series* if vertex b is on no other arc, and ab , ab are in *parallel*. In electrical terminology, for two resistors R_1 and R_2 in series, with the topological vertices a , b and c , this is:



And for two resistors R_1 and R_2 in parallel, with the topological vertices a and b , this is:



He then listed four types of operations on graphs:

- (1a) Replace an arc ab by two arcs ac and cb in series (c being a new vertex).
- (1b) Replace two arcs ac and cb in series by a single arc ab , dropping out vertex c .

In these operations a and b need not be distinct.

- (2) Break the graph at a single vertex into two connected pieces, or join two connected pieces at a single vertex.

By operations of this sort we can make one graph isomorphic with another if its components are respectively isomorphic with the components of the other.

(3) Suppose $G = H_1 + H_2$, where H_1 and H_2 have vertices a and b and no others in common, and a and b are connected in both H_1 and H_2 . If ac_1, ac_2, \dots, ac_m , and bd_1, bd_2, \dots, bd_n are the arcs of H_1 on a and b respectively (there is at least one arc in each set), replace these by the arcs $bc_1, bc_2, \dots, bc_m, ad_1, ad_2, \dots, ad_n$. We shall say simply, turn H_1 around at the vertices a and b .

From these he developed the relationships between two graphs if one is formed from another by using, at most the operations listed in the left-hand column below; with the relationship entered in the right-hand column:

no operations	isomorphic
(2)	1-isomorphic
(2) and (3)	2-isomorphic
(1a) and (1b)	homeomorphic
(1a), (1b) and (2)	1-homeomorphic
(1a), (1b), (2) and (3)	2-homeomorphic

Using these operations, Whitney proved a number of theorems classifying elementary graphs, according to their nullity and according to their rank. In the penultimate part of his paper, Whitney outlined a standard method for the construction of graphs, which he categorised as follows:

- (1) The basic graphs of nullities 1 and 2 are known. We form successively the basic graphs of nullities 3, 4, ... , as in Theorem 13.
- (2) From the basic graphs of a certain nullity, we form all elementary graphs of the same nullity as in Theorem 11. Any non-separable graph we can form thus is elementary (Theorem 10). A given elementary graph may, however, be derived from different basic graphs. If we wish, we can forget all graphs formed which have 2-circuits (arcs in parallel).
- (3) Taking all elementary graphs of a given nullity, we form all non-separable graphs of the same nullity by operations first of type (1a) and then of type (3) (Theorem 3). If we left out elementary graphs with 2-circuits and wish to include graphs with 2-circuits now, we must add arcs in parallel with various arcs of non-separable graphs with lesser nullity. We form finally any graph by taking non-separable graphs and letting vertices coalesce in such a manner that the graphs are the components of the final graph.

In the final part of the paper Whitney discussed classifying graphs by nullity.

Paper 9 Planar graphs

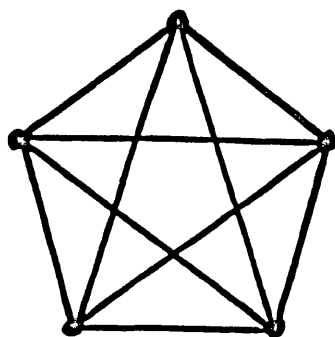
In this seminal paper [18] dealing with planar dual graphs Whitney gave a converse argument to that in his paper *Non-separable and planar graphs* [9] regarding the definition of planar graphs involving the graphs K_5 and $K_{3,3}$. It was the third paper that Whitney presented to the American Mathematical Society on 28 December 1931, but it was not published until 1933.

Kuratowski’s theorem stated that, in a topological way, a graph is planar if it does not contain either G_1 (K_5) or G_2 ($K_{3,3}$). Therefore, this argument may be used to provide an alternative proof that a graph that has a Whitney dual must be planar. In this current paper, Whitney reversed the argument by proving that a graph not containing K_5 or $K_{3,3}$ must have a Whitney dual.

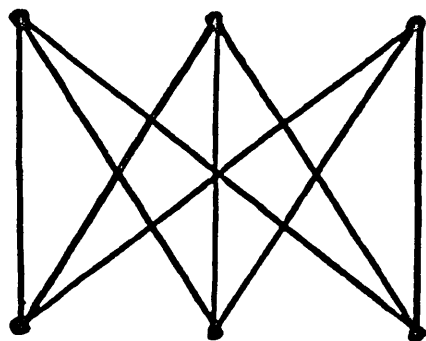
The main purpose of the paper was to give a proof of Kuratowski’s theorem:

Theorem 12. A necessary and sufficient condition that a graph have a dual is that it contain neither of the two following types of graphs as a subgraph:

- K_1 . This graph is formed by taking five vertices, and joining each two by an arc or suspended chain.
- K_2 . This graph is formed by taking two sets of three vertices each, and joining each vertex in one set to each vertex in the other set by an arc or suspended chain.



K_5 (Whitney’s K_1)
(Kuratowski’s G_1)



$K_{3,3}$ (Whitney’s K_2)
(Kuratowski’s G_2)

Towards the end of his paper, *Non-separable and planar graphs* [9] Whitney had included the following theorem, acknowledging that the theorem had been proved previously by Kuratowski [19]:

Theorem 31. A necessary and sufficient condition that a graph be planar is that it contain neither of the two following types of graphs as a subgraph:

G_1 . This graph is formed by taking five vertices a, b, c, d, e , and joining each pair by an arc or suspended chain.

G_2 . This graph is formed by taking two sets of three vertices, a, b, c , and d, e, f , and joining each vertex in one set to each vertex in the other set by an arc or suspended chain.

This alternative proof provided a combinatorial rather than a topological characterisation of planarity and of Kuratowski's theorem. Additionally Kuratowski's theorem implies that every non-planar graph contains at least one subgraph homeomorphic to K_5 or $K_{3,3}$. Homeomorphic is defined in [1] as:

If G is a given graph and H is a graph derived from it by subdividing some of the edges, then H is said to be a **subdivision** of G . Any two subdivisions of G are equivalent from a topological point of view, and they are said to be **homeomorphic** to each other, and to G .

Although Kuratowski's paper was published in 1930, his results had been submitted to the Polish Mathematical Society in June 1929. At the same time two other mathematicians had been independently carrying out similar work. They were the Americans Orrin Frink (1901–1988) and Paul Althaus Smith¹: an abstract of their paper was published in the *Bulletin of the American Mathematical Society* [20], indicating that it had been received on 10 February 1930, but as their paper was too similar in content and in its proof, it was rejected by the *Transactions* of the Society.

In his paper Whitney called upon many definitions, arguments, and theorems from two of his earlier papers [9] and [13], and from [21] (the next paper), some parts of which were used to prove Theorem 2. Theorem 4 defined a correspondence, using Theorem 9 from paper [9], in dual graphs between cycles in one graph and cutsets in the other. Theorem 9 deals with the relationship of vertices on a cycle in dual graphs with chains consisting of distinct vertices, whilst Theorem 11 gives conditions for preserving the correspondence between arcs in dual graphs.

¹ The author has been unable to ascertain his dates of birth and death, although an unconfirmed source gave his year of death as 1980.

In fact, Whitney describes the content of the paper thus:

In § 1 we show that, to determine whether two graphs are duals or not, it is not necessary to regard all their subgraphs, but merely a part of them. This section will not be used in the sequel. In § 2 cut sets of arcs are discussed. In §§ 3, 4, and 5 some properties of planar graphs are described which correspond to common point set theorems in the plane. The rest of the paper is devoted to the proof of Theorem 12. In this proof we need only Theorems 4, 9, and 11.

Paper 10 *2-isomorphic graphs*

On 30 August 1932 Whitney presented this paper [21], to the American Mathematical Society, and published it in the same issue of the *American Journal of Mathematics*, and directly followed his earlier paper [16]. In this new relatively short paper he made reference and use of the previous paper [18], although as previously noted this paper, presented in 1931, was not published until 1933 along with his papers [6, 9, 11, 13].

This paper contained a lengthy detailed proof of Whitney's theorem:

Theorem: If there is a 1–1 correspondence between the arcs of the two graphs G and G' so that circuits correspond to circuits, then the graphs are strictly 2-isomorphic.

This was described by Tutte as [22]:

This is Whitney's famous theorem that a 3-connected planar graph can be drawn on the sphere in essentially only one way.

Paper 11 *A numerical equivalent to the four color map problem*

On 26 December 1933, Whitney presented this paper [23] to the American Mathematical Society. It began with Whitney pointing out in the *Introduction* the 'essential difficulty of the problem' (referring to the four-colour problem) and going on to say that the difficulty is because the problem involves two disparate types of questions, described as:

What kind of a configuration is a map, and what kind of configuration can be colored in four colors; the first problem is geometric in character, and the second, combinatorial.

In this paper he established the method of studying the four-colour problem as a numerical problem. The intention of the paper was to develop a statement that is entirely combinatorial but which is equivalent to the four-colour conjecture.

In Section 2, *The Problems*, Whitney defined the problems:

I. A map is colored by assigning to each region a color in such a manner that no two regions with a common boundary are of the same color.

Problem I. Given any map on the surface of a sphere, is it always possible to color it in four colors?

II. Draw a regular polygon; divide the inside into triangles in any manner by drawing non-intersecting diagonals; divide the outside into circular triangles in any manner by drawing non-intersection arcs (which we also call diagonals). The result we shall call a polygonal configuration. It is colored by assigning to each vertex of the polygon a color in such a manner that two vertices which are joined by a line, either a side of the polygon or a diagonal, are of the same color.

Problem II. Can all polygonal configurations be colored in four colors?

III Let $(p_1, q_1), (p_2, q_2), \dots, (p_s, q_s)$ be pairs of integers. We shall say they form an n -admissible set of pairs if:

$$1 \leq p_i, p_i + 2 \leq q_i, q_i \leq n \quad (i = 1, \dots, s)$$

$$p_i < p_j < q_i < q_j \text{ is true for no } i \text{ and } j$$

We say the set of pairs is complete if it is impossible to add a new pair of integers in such a manner that the above relations still hold. Any two complete n -admissible sets have the same number of pairs.

Problem III. Let $(p_1, q_1), \dots, (p_s, q_s)$ and $(p'_1, q'_1), \dots, (p'_s, q'_s)$ be any two complete n -admissible sets of pairs. Is it always possible to find numbers b_0, \dots, b_n each equal to 1, 2, 3 or 4, so that

$$b_i \neq b_{i+1}, \quad (i = 0, \dots, n-1)$$

$$b_{p_i} \neq b_{q_i}, b_{p'_i} \neq b_{q'_i} \quad (i = 1, \dots, s)$$

IV. Consider the sum $a_1 + a_2 + \dots + a_n$. If we put in as many pairs of parentheses as possible, we will have defined a definite manner of evaluating the sum. Such a sum with parentheses we call an arranged sum. For instance, $a + b + c$ may be arranged as $(a + b) + c$ or $a + (b + c)$; $a + b + c + d$ may be arranged as $((a + b) + c) + d$, $(a + (b + c)) + d$, $(a + b) + (c + d)$, $a + ((b + c) + d)$, or $a + (b + (c + d))$. The number a_1, \dots, a_n are the terms of the sum. Any number formed in evaluating the sum is a partial sum. Thus the partial sum of $(3 + 7) + (1 + 4)$ are 3, 7, 10, 1, 4, 5, 15.

Problem IV. If an n -fold sum (any n) is expressed in any two ways as an arranged sum, is it always possible to choose the terms of the sum as integers so that no partial sum of either arranged sum is divisible by 4?

In particular, no term may be divisible by 4; we may choose each term as 1, 2 or 3.

He outlined his approach which was to define the four-colour conjecture as Problem I and then prove that Problem I is equivalent to his Problem II, then repeat the process for Problems II and III and finally for Problems III and IV.

Problem IV is concerned with arrangements. For example, for the sum $a_1 + a_2 + a_3 + a_4$ with brackets added is called an arranged sum and following the normal convention that they be so positioned that no more than two terms are added at a time. For example, $a_1 + a_2 + a_3 + a_4$ may be written as $\{[(a_1 + a_2) + a_3] + a_4\}$, with an example of a partial sum being $(a_1 + a_2)$.

The remainder of the paper covers the proofs of his equivalences, following which Whitney gave examples of colouring maps and detailed a calculation of the probability that there exists a map of more than N regions that cannot be coloured in four colours. He concluded that for N sufficiently large, this probability is very small, echoing the work done earlier by Heawood.

Paper 12 *On the abstract properties of linear dependence*

In this paper [24], Whitney introduced the concept of a *matroid*. The paper was presented to the American Mathematical Society in September 1934. In the foreword to a book by Tutte [25], D R Fulkerson later wrote:

Matroid theory began with Hassler Whitney's basic paper "On the abstract properties of linear dependence" ... and has since been developed most intensively by the present author, W T Tutte, who has obtained deep results in the theory. As the name suggests, a matroid is something like a matrix. The concept in fact generalizes that of "matrix"; in particular, a matroid may be regarded as a generalizing of a graph or network.

From his work on graph theory Whitney concluded that there are analogies between the concepts of independence and rank for graphs and those of linear independence and dimension in vector spaces. In this paper he introduced the idea of a matroid to provide a framework to explain these analogies. He gave a number of equivalent definitions of a matroid; the following one, in terms of ‘independent sets’ and given in the *Introduction*, reads:

Let us call a system obeying (a) and (b) a “matroid”. (a) and (b) being —

(a) Any subset of an independent set is independent

(b) If N_p and N_{p+1} are independent sets of p and $p + 1$ columns respectively, then N_p together with some column of N_{p+1} forms an independent set of $p + 1$ columns.

If E is a non-empty finite set, and if I is a non-empty collection of \mathcal{I} subsets of E fulfilling the above conditions (and called independent sets), then the matroid $M = (E, I)$ is said to be defined in terms of its independent sets. The rank $r(A)$ of a subset A of E is the number of elements in the largest independent set contained in A .

Matroids may also be defined in terms of circuits thereby generating properties of cycles in graphs. If E is a non-empty finite set and C is a non-empty collection of subsets of E fulfilling the postulates given below (and called circuits) the matroid $M = (E, C)$ is said to be defined in terms of its circuits. Whitney wrote:

Postulates for circuits. Let M be a set of elements and let each subset either be or not be a “circuit”.

We assume:

(C1) No proper subset of a circuit is a circuit.

(C2) If P_1 and P_2 are circuits, e_1 is in both P_1 and P_2 , and e_2 is in P_1 but not in P_2 , then there is a circuit P_3 in $P_1 + P_2$ containing e_2 but not e_1 .

This paper, although not initially recognised as seminal [26], became a significant contribution to the study of graph theory and other areas of combinatorics and was taken up by others in the 1950s, including Tutte (see Chapter 9).

These twelve papers, all written within five years, were an enormous achievement and an important advance in graph theory. Of these papers, Tutte later said [27]:

The graph-theoretical papers of Hassler Whitney, published in 1931–1933, would have made an excellent textbook in English had they been collected and published as such.

6.3 Saunders MacLane (1909–2005)

Around this time, another American mathematician, Sanders MacLane, who became a colleague of Birkhoff at Harvard and who knew and corresponded with Whitney, published three papers in graph theory. The first was entitled *Some unique separation theorems for graphs* [28] in 1935. This paper explores methods of uniquely defining the separation of graphs by chains (as defined by Whitney in his papers on planar graphs), and was presented to the American Mathematical Society on 7 September 1934. The paper mentions and builds on the work of R M Foster and Hassler Whitney. In the concluding paragraph MacLane states that the techniques he described can be used to study the separation of graphs by cycles which he did in his two later papers published in 1937. One was entitled *A combinatorial condition for planar graphs* [29] and the other *A structural characterization of planar combinatorial graphs* [30].

In his second paper [29], MacLane provided another condition for graphs to be planar. It was presented to the American Mathematical Society on 11 April 1936. He mentioned Kuratowski's condition for a topological graph to be planar, i. e., 'that it can be mapped in a 1–1 continuous manner on the plane, if and only if it contains no subgraph having either of two specific forms' [19] and Whitney's definition that a graph is planar if and only if it has a combinatorial dual [9]. MacLane's condition is contained in his Theorem I:

Theorem I. A combinatorial graph is planar if and only if the graph contains a complete set of circuits such that no arc appears in more than two of these circuits.

Paper [30] also dealt with planar graphs and with cycles. It was presented to the American Mathematical Society on 30 December 1936. The paper began by reiterating the

work of Kuratowski and Whitney and himself in the previous paper on the ‘known necessary and sufficient condition that a combinatorial graph be planar’. He continued by saying:

We seek an intrinsic condition; that is, a condition expressible in terms of configurations which are associated in a unique manner with a given graph.

The condition that MacLane developed was his Theorem 5:

Theorem 5: A set of circuits C_1, \dots, C_m in a non-separable graph G is the set of complementary domain boundaries of a planar map of G if and only if each edge of G is contained in exactly two of the circuits C , while the circuits C_1, \dots, C_{m-1} form a complete independent set of circuits in G , mod 2.

This implies that, for a non-separable graph to be planar, it must contain a set of cycles all of which have all their edges lying in exactly two of the cycles.

During the time he was an assistant professor at Harvard MacLane wrote his famous textbook *A Survey of Modern Algebra* [31] with Garrett Birkhoff, which provided a much needed undergraduate text. After World War II, and as Professor of Mathematics at Chicago University, MacLane worked on a wide range of mathematical topics including logic, planar graphs, cohomology, category theory, and he had a distinguished career in algebraic topology.

Conclusion

Like Veblen, Birkhoff and Franklin, Whitney had a long and successful career in mathematics and played a major part in the development of graph theory in America and of mathematics worldwide.

On many occasions, their work was inspired by and developed that of the others. Their academic careers were intertwined throughout the period covered. Veblen and Birkhoff had both been supervised by E H Moore at Chicago, and worked together at Princeton for a few years prior to the First World War, before going their separate ways. Even then, their

paths continued to cross. Veblen was responsible for supervising Franklin's doctoral research, and Birkhoff provided the same for Whitney. There are many instances in their collective papers where mention is made of the others, in some cases developing previous work and in others making corrections and/or improvements or simplifications. In their later graph theory papers, each of them (either individually or in collaboration), provided substantial publications, such as the collaborative paper of D C Lewis and G D Birkhoff (see Chapter 8) and that of H Whitney and W T Tutte (see Chapter 9) that summarised the state of graph theory at those times.

The story of graph theory in America continues in Chapter 8. The next chapter reflects on external influences that scholars experienced in the 1930s and World War II, and their impact on the mathematicians included in this narrative.

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Chapter 7

The 1930s and World War II

During the 1930s events in Europe, and particularly those involving Germany, had a considerable impact, not only on the world at large, but also on those employed in higher education in the USA. These happenings triggered an exodus of people from German-speaking countries including many Jewish mathematicians, because of persecution. Many went to America where they were at times subject to prejudice and anti-Semitism, but were also soon to be caught up in a new global conflict. Most actively and proudly did what they could to assist their new country and its allies in defeating German, and later Japanese, aggression.

7.1 Immigration

Throughout its history, the United States of America has been a haven for the oppressed of other countries. Therefore, it was no surprise that, large numbers of American citizens moved to help those who were subject to persecution in German-speaking countries, following the election of Adolf Hitler as Chancellor of Germany in 1933. These included many academics who could be supposed to live in their ivory towers of learning who became active in the efforts being made to help individual scholars that were losing their livelihoods because they were non-Aryans or politically unacceptable to the Nazis.

The Academic Assistance Council had already been formed in Britain, which influenced the efforts being made in America. More displaced scholars were eventually absorbed into the United Kingdom than into the USA, but that fact should not be seen as a criticism of the American people or government.

Two organisations were formed in America to plan the immigration and absorption of these displaced scholars. These were the Emergency Committee for Displaced German

Scholars (later Displaced Foreign Scholars) and the Rockefeller Foundation [1]. At the conclusion of World War II, around 120 to 150 mathematicians who had been dismissed from their posts by the Nazis had entered the USA and many remained permanently. It was by no means an easy task that these two agencies had set themselves, as America was still recovering from the 1929 stock market collapse and subsequent economic depression. Some in the academic world believed that an influx of foreign scholars would deprive home-grown talented young people from obtaining suitable positions. Additionally, there were the feelings of nationalism and anti-Semitism, with which to contend.

Foreign mathematicians benefited well not least because mathematicians and their associates were prominent in both organisations; for example, the president and head of the natural science programme of the Rockefeller Foundation were both mathematicians. In addition, many of the older mathematicians came from European families that had emigrated to the USA only one or two generations before and were of the age to remember and have experienced the days when American mathematicians travelled to Europe for postgraduate study, and by so doing established close academic and personnel relationships with European Universities, many of which were in Germany, then the leading centre of mathematics.

To give an example of the effects of the depression, by October 1933, 2000 out of 27000 teachers had been dismissed from the faculties of 240 American institutions of higher education, as reported by Edward R Murrow, the second-in-command of the Emergency Committee [1]. The Committee and the Foundation decided that they would use their funds to aid scholarships, and endeavour not to displace existing faculty members, nor act such that their discussions should encourage anti-Semitism or resentment of incoming foreigners. Both organisations wanted to place the immigrants into short-term funded research posts, anticipating that this would lead to permanent positions, rather than into teaching. Even with this philosophy, their actions were not without critics within the

American education system, with some native scholars resentful that foreigners were given the opportunity to carry out research while they, with their heavy teaching loads and lack of funds, were not able to do likewise. Additionally, some felt that the influx would deny young home-grown scholars from progressing up the education hierarchy.

One of the foremost mathematicians actively involved in procuring the entry into the USA of displaced scholars from Nazi persecution was Oswald Veblen. Veblen was not a timid man [2], but rather someone who wanted fair play and opportunity for everyone. For example, in 1943 he declined to fill in the entry for 'race' on a form associated with his war work at the Army's Ordnance Aberdeen Proving Grounds; after all, he had been a major in Ordnance working on ballistics research during World War I. He sent a letter of protest to the Secretary of War saying that the question was invidious and the sort that would have been the norm in Germany at that time. In 1946 he again refused to sign a form which waived the right to strike at Aberdeen, and a few years later, during the infamous McCarthy witch-hunt period, it was suggested, falsely, that he was a communist, and should be denied a passport. He was not a communist: at the time, he claimed to be an old-fashioned liberal.

Veblen became a member of the Emergency Committee at its foundation and provided detailed information on each possible immigrant. Along with Hermann Weyl (1885–1955), he ran a placement bureau for displaced mathematicians until the end of the war. As described by Nathan Reingold [2]:

In Veblen's papers in the Library of Congress are lists of names with headings such as scholarship, personality, adaptability and teaching ability. When information about a person was incomplete in the United States, Veblen wrote to European colleagues.

In 1933, the American Mathematical Society formed a committee to cooperate with the Emergency Committee. Veblen was one of the three men appointed to the new committee.

As mentioned earlier, Germany was the leading centre for mathematics in the period before 1933 and, in the years before the Nazi party came to power, published the most

respected of reviewing and abstracting vehicles, the *Zentralblatt für Mathematik*. The journal's editor was Otto Neugebauer (1899–1990) who, whilst not being Jewish, held views that were politically unacceptable to the new German regime, and was forced to flee to Denmark in 1934. In 1938, Tullio Levi-Civita (1873–1941), an Italian mathematician, was not only dismissed from his professorship, but also removed from the board of *Zentralblatt* for political reasons. A number of resignations swiftly followed, including G. H. Hardy (1877–1947), Harald Bohr (1887–1951), and Oswald Veblen. The journal had also decided that refugee mathematicians and Russians should be barred from being collaborators and referees for its published papers and articles. The reaction in the USA and elsewhere was one of indignation that the independence of scientific internationalism had been violated, and that the worldwide body of mathematicians had been insulted and its integrity impugned.

As a result of the actions of *Zentralblatt für Mathematik*'s governing board and the resignations of advisory members Veblen urged that American mathematicians found a new review journal, a suggestion that he had made some fifteen years before, although he recognised that at that time the mathematics community was not ready to take on the task. However, in 1938 Veblen believed that such a move was possible in the USA: possible, because the number of mathematicians carrying out research had considerably increased over the previous two decades, including the recent immigrants, and that America was now becoming the world centre for mathematics. In December 1938, the American Mathematical Society formed a committee and all the different factions of the American mathematics community joined the discussions. The resulting discussions covered political, religious and racial questions, and considered financial security and international cooperation. Veblen used his considerable talent of persuasion to obtain a \$65,000 grant from the Carnegie Corporation. After much debate and the propounding of many conflicting opinions, not least the pro-German stance by the mathematics department of

Harvard (including G. D. Birkhoff), the council of the Society voted on 25 May 1939 for the journal. The voting was 22 for, 5 against, and 4 uncertain. Oswald Veblen was appointed chairman of the committee entrusted to organise and launch the new publication. The publication was called *Mathematical Reviews*.

The two organisations helping the dispossessed and politically unacceptable academics to enter America had as their policy the restrictions that the immigrants must be eminent and that they be placed in institutions with research capabilities. Veblen, Weyl, and others were soon helping the non-eminent and placing the refugees in other universities and colleges, including junior colleges that would take them. This unofficial policy was controversial and did not find agreement in some quarters, particularly Harvard, which endeavoured to raise other funds for aiding refugees in an attempt to ease the Emergency Committee aside and regain control of faculty appointments. To avoid a major crisis within the mathematics fraternity, Veblen sought an agreement with Birkhoff. On 24 May 1939 Harlow Shapley, head of the Harvard College Observatory, wrote [3]:

When Veblen and Birkhoff were in my office the other day, it was agreed that the distribution of these first rate and second rate men among smaller American institutions would in the long run be very advantageous, providing at the same time we defended not too feebly the inherent right of our own graduate students.

This agreement was most probably reached because both men, and others in the American Mathematical Society, believed that the USA would soon be at war, at which time mathematicians would be in short supply so that the newcomers would be welcome.

Many pure mathematicians believed that the application of mathematics was the responsibility of other disciplines, such as physics, chemistry and engineering, an attitude that held back the advancement of applied mathematics. They believed that the role of mathematics faculties throughout the USA was to devote themselves to research and development of mathematics for mathematics' sake. However, as in the Great War, the

standing of applied mathematics was to be considerably advanced as a result of the necessities of World War II, which was assisted by the new immigrants.

After the war, the USA could be reasonably proud that, for all the entire pro- and anti-opinions and the many arguments and deals, the nation, once again, had become a haven for the oppressed when it was required.

7.2 The Second World War

As in the first global conflict, most patriotic citizens in the USA willingly contributed their skills to the winning of peace and, as before, several hundred mathematicians were among them. Oswald Veblen returned to the Ballistics Research Laboratory at the Aberdeen Proving Grounds, taking responsibility for recruiting suitable mathematicians. During this task, when confronted by reluctance from a university or college to release a suitable candidate, Veblen arranged for an ‘alien’ who was not able to work in a military establishment to substitute; one such ‘alien’ was his own assistant at the Institute for Advanced Study, Gerhard Karl Kalisch. As mentioned above, the influx of talented mathematicians went to assist only in providing the numbers required for war work and the continuation of the education of the nation.

Again, as in 1917, some mathematicians became uniformed combatants, some as both enlisted and civilian staff, joining units such as the Army, Navy, and Air Bureaux of Ordnance, while others remained in their academic posts but provided much needed training programmes, and some went into industry to work on war-related projects. A number were recruited into cryptanalysis and into the Manhattan Project that had been set up to develop the atomic bomb. Before the war, applied mathematics did not feature highly in mathematical departments in universities and colleges in the USA; its development was mostly left to engineers and physicists. This state of affairs still pertained, even though it replicated the situation of America’s entry into the Great War in 1917. Again, as before,

those professional mathematicians engaged in applied mathematics were looked down upon by some pure mathematicians, as it was thought by many that applied mathematics practitioners found pure mathematics too exacting. That is as may be, but the time had come again when all those involved in the war effort were required to tackle whatever task was assigned to them, and most of these were in the realm of applied mathematics. In 1942, the Applied Mathematics Panel was formed within the National Defence Research Committee. The Panel's policy was developed by the Committee Advisory to the Scientific Office whose membership included Veblen. The Applied Mathematics Panel was the largest group of mathematicians under the wartime government and provided assistance wherever it was needed, from 1942 to the end of the war. In many cases the work undertaken by these scholars did not require a high level of mathematics, but because of their undoubted ability to apply themselves to problems in a logical and analytical manner, they provided fast and penetrating solutions.

Within the Ballistics Research Laboratory, the computing group were responsible for the construction of artillery firing tables and for level-bombing tables. Another group, variously known as the math unit, the math section and the theory section, developed computational procedures for ballistics firing and undertook troubleshooting for a wide range of varied projects. Additionally, they carried out emergency work on a variety of projects including the investigation of the causes of malfunctioning ordnance.

A number of new fields of learning, most having connections with mathematics, were developed because of the requirements of the military. These included the development of Operations Research that after the war was one of its prominent fields, and linear programming was started as a natural progression from the Air Force planning activities. Other areas were statistics and probability, digital computers and their associated sciences, sequential analysis, and cryptanalysis.

As has been seen in the previous chapter, Veblen was not the only mathematician who made significant contributions to ordnance during the Second World War. Hassler Whitney was a member of the group at the Aberdeen Proving Grounds and received awards for his work. As with the First World War, no record of G D Birkhoff assisting in the second worldwide conflict could be found, even though he was 4 years younger than Veblen.

7.3 Anti-Semitism

Just like other forms of prejudice, anti-Semitism has existed for millennia. It has waxed and waned over time and moved geographically around the globe. Anti-Semitism was to be found within American society between the two world wars and, of course, it was practised by some mathematicians of the time in the environs of the senior places of higher education. Prior to the general understanding of Hitler and Adolf Eichmann's 'final solution' and the emergence of the proof of the holocaust, anti-Semitism was seen, not as mortal sin, but as ugly and petty minded.

It is difficult to differentiate sometimes between nationalistic feelings and anti-Semitism. Some expressions of national sentiment have no hidden anti-Semitic intent (in fact, no anti-foreigner meaning whatsoever), but unfortunately some comments in favour of nationalism and against aliens mask prejudices. Although the history of the USA has been built on immigration, many recently arrived newcomers expressed an antipathy towards other foreigners, especially those also wishing to settle in America. Because many of the immigrants fleeing Nazi Germany were of Jewish extraction, it was easy to hide anti-Semitic prejudice behind concerns that the incoming scholars would deprive home-grown postgraduates from securing appropriate positions.

In the 1920s and 1930s there were many examples of university faculties operating an anti-Jewish policy, and those who allowed the employment of a Jewish scholar rarely hired

a second. However, there were many in mathematical departments who showed no anti-Semitism and positively worked to undermine the actions of their colleagues who did.

As mentioned in Chapter 4, George David Birkhoff showed anti-Semitic tendencies and although perhaps others were even more so, it is because he was then one of the two leading mathematicians in the United States of America that his words and actions are remembered, especially as the other leading mathematician, Oswald Veblen, was very much the opposite. Birkhoff was initially very much against Solomon Lefschetz (1884–1942) becoming the first Jewish president of the American Mathematical Society, and a letter from him to R G D Richardson, Secretary of the American Mathematical Society from 1921 to 1940, included [4]:

I have a feeling that Lefschetz will be likely to be less pleasant even than he had been, in that from now on he will try to work strongly and positively for his own race. They are exceedingly confident of their own power and influence in the good old USA. The real hope in our mathematical situation is that we will be able to be fair to our own kind ...He will get very cocky, very racial and use the Annals [Annals of Mathematics] as a good deal of racial perquisite. The racial interests will get deeper as Einstein's and all of them do.

At the American Mathematical Society semi-centennial meeting in 1938, Birkhoff presented a historical survey of mathematics in the USA over the previous fifty years. In it, he discussed foreign-born mathematicians, particularly the immigrant scholars of the 1930s, who (he believed) had an unfair advantage over the native born. Although the list of names contained in his address included those who were neither German nor Jewish, many of his audience considered Birkhoff's views anti-Semitic, causing much heated debate long after the occasion.

Lipman Bers (1914–1993) in the section of his article *The Migration of European Mathematicians to America*, entitled *Anti-Semitism*, states that Birkhoff 'was not free from anti-Jewish prejudices', and wrote [5]:

The same Birkhoff who could toss off an anti-Semitic remark in a private letter, did not let his racial prejudices interfere with his evaluations of other peoples' scientific work. The late complex analyst

Wladimir Seidel, who graduated from Harvard and later taught there as a Benjamin Peirce Instructor, told me about a phone call made by Birkhoff to a departmental chairman. “I know you hesitate to appoint the man I recommended because he is a Jew. Who do you think you are, Harvard? Appoint Seidel, or you will never get a Harvard PhD on your faculty.” Seidel was duly appointed.

Conclusion

That period of our history was a desperate time, but most of those who lived through it were determined that it should not happen again. When the USA entered World War II at the end of 1941, Hassler Whitney was already a potent force within the world of mathematics. He was thirty-four years of age, an associate professor at Harvard University, and had already published most of his seminal work on graph theory. In the 1940s other scholars in America and around the world were to take up the challenges in graph theory laid down by Whitney and his American predecessors. They were to continue to develop the subject, particularly that part concerned with the four-colour problem. Even the outbreak of the second global conflict did not prevent research into graph theory. A little of what was achieved is explored in the next chapter.

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Chapter 8

American graph theory in the 1940s

Although many mathematicians in the United States of America and Europe were serving their countries by contributing to the war effort, they also continued their research. The scholars publishing results in graph theory during the 1940s included the Americans Birkhoff, Daniel Clark Lewis Jr., and Arthur Bernhart.

8.1 Daniel Clark Lewis (1904–1997)

During the last few years of his life, Birkhoff collaborated with D. C. Lewis in an attempt to bring together all the previous quantitative work on map colouring, and to offer some of their own conjectures.

Lewis was born in New Jersey and obtained his PhD in 1932 under the supervision of Birkhoff at Harvard University. He worked from 1943 to 1945 at the War Research Establishment at Columbia, after which he taught at Johns Hopkins University, and later in 1958–1959, he carried out research at the Institute for Advanced Study. One of his main contributions to mathematics was the discovery and development of the theory of ‘autosynartetic’ solutions, generalising a theory of Poincaré on periodic solutions of ordinary differential equations.

In the course of their working together, Birkhoff and Lewis were aware of the possibility that Reynolds may have developed a solution to the four-colour problem, a suggestion that Reynolds made to Lewis in a telegram in August 1942 [1]. Correspondence between Birkhoff and Lewis suggests that they were unsure whether Reynolds had achieved what he claimed. In addition, Reynolds’ claim utilised some work of Birkhoff and Lewis that had been presented at a meeting of the American Academy of Arts and

Sciences. Because of his concerns, Lewis endeavoured to persuade Birkhoff that they should publish some of their joint work [1]:

If Reynolds really has solved the problem (I remain sceptical until I have chance to see what he has actually done), don't you think it would be a good thing for us to publish immediately the part of our work on which he based his solutions?

A few days afterwards, they received a copy of Reynolds' work and found that the claim was false. Lewis wrote to Reynolds gently pointing out where the proof was unconvincing, indicating that if the work were published, neither he nor Birkhoff wished to take credit for any work upon which his 'proof' was based. On 1 September 1942, Reynolds wrote to Birkhoff [2] (and most likely also to Lewis):

Dear Professor Birkhoff

Referring to my two communications concerning the four color problem which have been sent to you this summer, you will please carefully place the second one in your waste basket.

Yes, I have burned my fingers!

During the forty eight hours following my receipt of Professor Lewis's communication pointing out my out my [sic] error I was firmly resolved never to touch the problem again.

But I've already fallen from the top of the four color water wagon. I'm at it again! Seriously, however.

I do sincerely apologise for sending you my last letter.

With a very red face *.

Clarence N Reynolds

*Customary among those who fall from the water wagons!

Birkhoff also wrote to Reynolds, and a letter to Birkhoff from Lewis included [3]:

I recently had a letter from Reynolds. He's still plugging away at the problem apparently! He said he got a nice letter from you saying that he wasn't the only man who had stumbled by the mathematical way side!

One can empathise with Reynolds for finding out that what he thought was a solution to one of the world's most famous problems was not so, and for having to swallow his pride before one of the most world-famous and accomplished mathematicians.

The working alliance between Birkhoff and Lewis resulted in a lengthy paper entitled *Chromatic polynomials* [4], which was presented by Lewis to the American Mathematical Society on 23 August 1946, two years after Birkhoff's death.

The paper, which included most of Birkhoff's work on map colouring, was 97 pages long, and from the correspondence between the two scholars, it appears that Lewis had been responsible for its writing. Indeed, in a letter dated 21 August 1942 [5] he mentions, 'I have so far written some 90 pages. There will probably be about 20 or 30 pages more'. He was waiting for Birkhoff to return from South America so that the draft could be reviewed, and the letter indicated that he believed the final paper would be 'about 75 printed pages'. Towards the end of the correspondence, in early 1943, Lewis was clearly in control of the production of the paper and was relying on Birkhoff only for reviewing the work and making minor suggestions and comments. Although essentially completed in mid-1943, it was not until November 1945 that the Society received the manuscript. For many years, this work would be an authoritative resource on chromatic polynomials, and many later authors were to refer to it in their own papers.

The first part of the *Introduction* to the paper reads, 'Relation of the present work to previous researches on map-coloring and summary of results', which succinctly set the scene. It continues by defining two 'quite different types of investigation': Type 1 is the qualitative approach and Type 2 is the quantitative approach.

The work of the first two chapters is essentially quantitative. Chapter I gives the basic principles of chromatic polynomials, including the main results, and the proof of the following theorems:

Theorem I. Let T be an m -gon in a map P_n of n regions. Let $\Pi_{n-k}(\lambda)$ denote the sum of the chromatic polynomials associated with the sub-maps obtained by erasing just k boundary lines of T . Then:

$$P_n(\lambda) = \frac{1}{m} \sum_{k=1}^{[m/2]} (k\lambda - m) \Pi_{n-k}(\lambda),$$

where $[m/2] = m/2$ or $(m-1)/2$, according as m is even or odd.

Theorem II. Let the boundaries of T be denoted by I_1, I_2, \dots, I_m in cyclic order. Let $U'_{n-k}(\lambda)$ denote the sum of the chromatic polynomials associated with the sub-maps obtained by erasing the boundary I_i , together with just $k-1$ other boundaries of T . Then, with the notation of the previous theorem and for any positive integral value of i from 1 to m , inclusive:

$$P_n(\lambda) = (\lambda - m) \sum_{k=1}^{[m/2]} U'_{n-k}(\lambda) + \sum_{k=2}^{[m/2]} (k-1) U_{n-k}(\lambda).$$

Birkhoff used the letter P when talking about cubic and referred to Q for all other maps.

Chapter II covers the computation of chromatic polynomials. After defining the method used, they present a table of 111 chromatic polynomials, for all regular maps of 6 to 17 countries. Each regular map is designated by a symbol $(n; a, b, c \dots)$, where:

n = the total number of regions in the map

a = the number of four-sided regions in the map

b = the number of five-sided regions in the map

c = the number of six-sided regions in the map

and so on.

As an example, the chromatic polynomial for the nine-region map $(9; 5, 2, 2)$, divided by $\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$ is given as

$$Q_9(u) = u^5 + 0u^4 + 5u^3 + 0u^2 + 2u + 1, \text{ where } u = \lambda - 3.$$

The remainder of the chapter explored special results for other regular and non-regular maps.

Chapters III and IV of the Birkhoff–Lewis paper include many proofs of inequalities that are satisfied by the coefficients of chromatic polynomials. Also, in Chapter III, there is Birkhoff's 1912 determinant formula for chromatic polynomials. Throughout the chapter, much use is made of Whitney's Theorem II from in his 1930 paper *A theorem on graphs*. This theorem deduced (subject to defined restrictions) that a closed path can be drawn which passes through each region of the map once only, but does not pass through any vertex (a dual of a Hamiltonian path). In addition, Chapter IV contains the Birkhoff–Lewis,

conjecture, which they describe as stronger than the four-colour conjecture, and hope that it will be 'easier to establish'; Birkhoff believed that the conjecture would be of considerable significance in the eventual solution to the four-colour problem. The conjecture is that the following theorem always holds for cubic maps:

If P_{n+3} has a proper 2-ring or a proper 3-ring or a four-sided region K surrounded by a proper 4-ring, then

$$(\lambda-3)^n \ll Q_n(\lambda) \ll (\lambda-2)^n \quad \text{for } \lambda \geq 4,$$

provided that the same relation, with n replaced by m , holds for certain maps P_{m+3} with $m < n$.

The paper explained the use of \ll and \gg as:

Let $f(\zeta)$ and $g(\zeta)$ be two polynomials in ζ . Then (contrary to the notation of the preceding chapter) we shall write $f(\zeta) \ll g(\zeta)$ or $g(\zeta) \gg f(\zeta)$ if, and only if, the coefficients of $f(\zeta)$ are non-negative and not greater than the corresponding coefficients of $g(\zeta)$.

Here, Q_n is a map of n regions and $Q_n(\lambda)$ is its chromatic polynomial, then Q_n can be coloured in the λ given colours. They used experimental data from Chapter III to propose the conjecture and were able to verify that the formula holds when $n < 14$ (that is, for maps with fewer than 17 regions).

The final two chapters covered the analysis of rings (4-, 5-, 6-, 7-, and n -rings) using Kempe chains. Of these chapters the authors wrote:

This paper belongs primarily to Type 2. Its primary object is the study of the chromatic polynomials. Nevertheless in the later chapters (V and VI) the most characteristic method of Type 1, namely, that of the Kempe chains, has been taken over and modified so as to yield quantitative results in any number of colors. Simultaneously, on the other hand, we have gained by an alternative method a deeper insight into the nature of the results previously obtained only by investigations of Type 1 by use of the Kempe chains. This is true to the extent that we are now able, without using Kempe chains, to prove the reducibility of the following configurations which are fundamental in investigations of Type 1:

1. The four-ring (Birkhoff [1, p. 120]). [6]
2. The five-ring surrounding more than a single region (Birkhoff [1, pp. 120–122]). [6]
3. Four pentagons abutting a single boundary (Birkhoff [1, p. 126]). [6]

4. A boundary of a hexagon abutting three pentagons (Franklin [1, p. 229]). [7]

The authors go on to say that ‘Undoubtedly, numerous other similar configurations can be proved to be reducible by the same methods’.

However, the paper is not easy to read. W. T. Tutte (see Chapter 9), who took on the mantle of the world’s foremost graph theorist after the Second World War, referred to this paper, and in particular to the Birkhoff–Lewis theory, writing [8]:

They do give a partial theory of these equations in their paper, but I must confess I was never able to read right through it and understand it clearly.

8.2 Arthur Bernhart (1908–1989)

Arthur Bernhart, an American mathematician, joined the mathematics faculty of the University of Oklahoma in 1943, where he remained until his retirement. He published papers on geometry and was an early authority on curves of pursuit, but was also captivated by the four-colour problem. In his significant and very technical 1947 paper entitled *Six rings in minimal five-color maps* [9], he reviewed the then-known theorems on 4- and 5-rings and progressed to a thorough investigation of 6-rings, thereby extending Birkhoff’s work on reducibility and completing Birkhoff’s study of rings of six countries. As Franklin said in his review of the paper [10]:

It is shown that if 6-rings exist in minimal maps, the structure must be one of six types.

These six types, which Bernhart calls solutions, include:

Solution #1; Corresponding to a single hexagon inside.

Solution #2; Corresponding to two pentagons inside with a common edge.

Solution #3; Corresponding to three pentagons inside with a common vertex inside.

This paper, which was a thorough study of the reducibility of configurations of ring size 6, prompted, the following tribute [11] that refers to the eventual solution to the four-colour problem:

These results, together with those of Birkhoff, form the foundation of the work of Appel and Haken.

In another paper, published in 1948, entitled *Another reducible edge configuration* [12], Bernhart used an argument involving Kempe chains to prove that, for a regular map, the configuration consisting of an edge common to two hexagons which borders two pentagons is reducible for colouring with four colours.

Conclusion

By the middle of the twentieth century, considerable progress had been made on approaches to solving the four-colour problem. However, although many following in the footsteps of Birkhoff had arrived at numerous reducible configurations, with notable contributions from Franklin, Errera and Winn, little progress had been made on constructing unavoidable sets of configurations. Only Wernicke in 1904 (see Chapter 3), Franklin in 1922 (see Chapter 5), and Henri Lebesgue in 1940 [13] had made significant discoveries in this area. In his paper Lebesgue used the counting formula and Euler's formula to create some new unavoidable sets. Graph theory as a whole had also benefitted from the significant contributions from citizens of the USA, both native-born and naturalised.

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Chapter 9

In conclusion

Graph theory continued to develop after 1950, and the four-colour theorem was proved in the 1970s. Work continued in the USA, carried out by a new generation of graph theorists, many of whom were inspired by and developed the work of Veblen, Birkhoff, Franklin and Whitney. In fact, the later scholars, in a few instances, worked with these four major mathematicians.

There have been occurrences of masters and pupils cooperating in research work that extends the academic life of the master and enhances the work of the pupil. One example of this was the joint work of George Birkhoff and Daniel Lewis during the Second World War. Another fruitful collaboration was *Kempe chains and the four-colour problem* [1], written jointly by Hassler Whitney and the English-born naturalised Canadian, William T. Tutte; the latter was at the time the world's foremost graph theorist, as Whitney had once been in the 1930s. The paper was mostly written by Tutte.

Other scholars who made their mark in graph theory after 1950 were Frank Harary, who collaborated with many scholars, including Tutte; Gerhard Ringel and Ted Youngs, who proved the Heawood conjecture; Oystein Ore and Joel Stemple also collaborated and extended the Birkhoff number to 41; Frank Bernhart; and Kenneth Appel and Wolfgang Haken, who finally solved the four-colour problem.

9.1 William Thomas Tutte (1917–2002)

Bill Tutte was world renowned for two things: his groundbreaking work in graph theory and combinatorics, and his considerable contribution at Bletchley Park to the Allies' victory in the Second World War. He was also a very shy and modest man.

In 1935, Tutte entered Trinity College, Cambridge, to study Natural Sciences, specialising in Chemistry and graduating with a first-class honours degree. Shortly after starting his postgraduate study in chemistry, coinciding with the outbreak of the Second World War, he was asked by his tutor to offer his services to the highly secret organisation at Bletchley Park. Bletchley Park housed the wartime headquarters of the British Government Code and Cipher School, where German airborne communications and those of the Italians and Japanese were monitored. Tutte joined the unit in January 1941 and remained there for the duration of the war, producing work that was bettered only by Alan Turing (1912–1954) and belatedly earning glowing testaments — including that in the citation at his induction as an Officer of the Order of Canada, which included ‘As a young mathematician and codebreaker, he deciphered a series of German military encryption codes known as FISH. This has been described as one of the greatest intellectual feats of World War II’.

Tutte returned to academic life in Cambridge after the war, where he was elected to a research fellowship in mathematics at Trinity College. His thesis on *An Algebraic Theory of Graphs* earned him his PhD in 1948. It combined ideas from algebra and combinatorics to develop matroid theory. As has been seen in Chapter 6, matroid theory originates from an original paper by Whitney in 1935 and was a subject that would be considerably developed by Tutte. Tutte’s thesis gave rise to papers that put him at the forefront of graph theory, with him as its main proponent.

After receiving his doctorate he was invited by H S M Coxeter to join the University of Toronto in Canada, first as a lecturer and then as an Associate Professor. During his fourteen years there he became well known in mathematical circles around the world for his work in combinatorics, publishing around two dozen papers on the subject — including ones on planar graphs, factors of graphs, finite graphs, embedding of linear graphs in surfaces, chromatic polynomials, cubic graphs, and matroids.

In 1958 he was elected a Fellow of the Royal Society of Canada, and in 1962 he was appointed Professor of Mathematics at the recently opened University of Waterloo, Ontario, a position he held until his retirement in 1985 when he was made Professor Emeritus. The University had been founded in 1957 and was in the process of establishing its identity and standing within the academic world when Tutte arrived there. The University created around him what was to become its world-famous Department of Combinatorics and Optimization. In addition, Tutte and the University were influential in the setting-up of the *Journal of Combinatorial Theory*. Tutte was a prolific writer of papers and books, but his style of writing made parts of his publications difficult to understand.

In planarity he produced a seminal paper in 1959 [2] in which he developed the work of Whitney and included a proof of:

... a necessary and sufficient condition, in terms of matroid structure, for a given matroid M to be graphic (cographic), that is the bond-matroid (circuit-matroid) of some finite graph.

The condition was that M be regular and not contain the cycle-matroid of a Kuratowski graph. A regular matroid is the matroid of a regular chain-group, and a regular chain-group is an integral chain-group in which every elementary chain is an integral multiple of a primitive chain, where a primitive chain is one whose coefficients are restricted to the values 1, -1 and 0. A chain-group is a class of chains that is closed under the operations of addition and multiplication by an element of a commutative ring with a unit element and no divisors of zero.

In his study of factorisation Tutte published two major papers [3], [4]: the first of these was reviewed by H S M Coxeter, a colleague and the person who, as mentioned earlier, was responsible for Tutte being at the University of Toronto. The review [5] began:

A graph N of even order n is said to be prime if it contains no set of $\frac{1}{2}n$ branches which together use up all the nodes a_1, a_2, \dots, a_n . Let S denote a subset consisting of f of these n nodes. If we suppress the nodes S and all branches belonging to them, what is left of N will, in general, consist of several

disconnected pieces. Let h_u denote the number of these pieces that are of odd order. The author proves that the graph N is prime if and only if it contains an S such that $h_u > f$.

The second paper, on infinite graphs, explored the question — under what conditions does a locally finite graph contain a regular subgraph that includes all the vertices of the graph?

Tutte expanded the work of Whitney in investigating Hamiltonian cycles. Whereas Whitney had developed theorems for such cycles in maximal planar graphs, Tutte extended this work to general planar graphs, in three papers in 1946, 1956 and 1960 [6], [7], [8]. Although short, his paper entitled *On Hamiltonian circuits* [6] was significant in that he provided a counterexample that disproved Tait's conjecture that each cubic map contains a Hamiltonian circuit. Tutte's example was a regular spherical map with 25 regions, 69 edges and 46 vertices.

He also studied chromatic polynomials and the golden ratio τ ($= \frac{1}{2}(1 + \sqrt{5}) = 1.618034\dots$). In Chapter 6 it was shown that Birkhoff developed the chromatic polynomial $P(\lambda)$, which he had hoped would lead to a solution of the four-colour problem. After Birkhoff's death more work was carried out on chromatic polynomials and it was established that if a map is sufficiently large, it could generally not be coloured with one, two, or three colours, meaning $P(1)$, $P(2)$ and $P(3)$ are all zero, and implying that at least four colours must be necessary to colour a map in the required form and $P(4) > 0$. Mathematicians then looked for other values of x for which $P(x)$ is zero, as if they were found this would add to the understanding of chromatic polynomials. To this end, Tutte investigated the zeros of chromatic polynomials of planar graphs and found them near the value of the golden ratio and other related values. In 1969 he published two papers [9], [10] on this, the first with G Berman, and in 1970 he expanded on this topic with two further papers [11], [12].

9.2 Kempe chains and the four-colour problem

This paper, published in 1972 and written jointly by Whitney and Tutte, was prompted in part by a rumour that circulated in 1971 that the four-colour problem had been solved. This rumour arose from a claim by Yoshio Shimamoto (b. 1926) that he had compiled a proof based on the work of Heinrich Heesch (1906–1995), who had spent a good number of years studying the reducibility of maps and who had developed the *method of discharging* which was central to the unavoidability part of the eventual proof of the conjecture. The method of discharging is used to prove that a given set of configurations is an unavoidable set: each k -sided region in a map is assigned a ‘charge’ of $6 - k$, and the charges are redistributed from each region to its neighbours in such a way that the overall charge of the map is unchanged.

Heesch had coined the term *D-reducibility* for the reducibility of some configurations of countries and had developed a method for testing this property. Shimamoto’s approach was to assume that the four-colour conjecture was false. He demonstrated that there had to be a non-colourable map M that contained a configuration H that passed Heesch’s computer test for *D-reducibility*. He then developed a contradiction by showing that the *D-reducibility* of H implied that M could be coloured with only four colours. This ‘proof’ relied on considerable computer time and output, which did not inspire Whitney and Tutte: in the *Introduction* to their paper, they wrote:

This method of proof was greeted by the present authors (independently) first with some misgivings and then with real scepticism. It seemed to both of us that if the proof was valid it implied the existence of a much simpler proof (to be obtained by confining one’s attention to one small part of M), and that this simpler proof would be so simple that its existence was incredible. The present paper is essentially the result of our attempts to give a proper mathematical form to our objection.

The pair went on to say that they could find no ‘flaw in Shimamoto’s reasoning’, but decided that the computer work must be in error. In the end, they satisfied themselves that Shimamoto had not proved the four-colour conjecture, but had developed a construction for

D -irreducible configurations — however, the crucial configuration in Shimamoto's argument turned out not to be D -reducible, so the problem remained unsolved.

In their paper, Whitney and Tutte set out to give a broad explanation for an approach to the four-colour problem, to define Kempe chains, and to indicate how these could (or could not) be used. Their intention was to clarify the situation that existed in 1972, à propos of the four-colour conjecture, for themselves but also for others. The paper was treated with the respect that its two very distinguished authors warranted, and demonstrated that graph theory in the Americas was in safe hands.

With regard to the four-color problem, Tutte, in his 1974 paper *Map-coloring problems and chromatic polynomials* [14], ventured the judgment that only an optimist could conclude the possibility of an unavoidable set of reducible configurations with 'only 8000 elements'; this was a reference to the work of Heesch. Tutte's 1975 paper [15] was unenthusiastic about a possible computer-based solution to the four-colour problem, as he believed that a quantitative rather than a qualitative approach to the problem was necessary. However his reaction when the eventual solution was announced in 1976 was generous and is best summed up in the book *Four Colours Suffice* [16]:

But when Tutte heard the news he waxed eloquent, comparing their achievement with the slaying of a fabled Norwegian sea monster:

Wolfgang Haken
Smote the Kraken
One! Two! Three! Four!
Quote he: 'The monster is no more'

And when Tutte was interviewed by the press, he told them, 'If they say they've done it, I have no doubt that they've done it.'

Tutte was rightly acknowledged as the post-war leader of combinatorial thinking, and although not a citizen of the USA he was a worthy North American successor to Veblen, Birkhoff, Franklin and Whitney.

9.3 Frank Harary (1921–2005)

Another American collaborator was Frank Harary; although he did not confine his papers to mathematics he applied graph theory to many topics, especially in the social sciences. Among the areas of his study were the connections between graph theory and anthropology, biology, chemistry, computer science, geography, linguistics, music, physics, political science, psychology and social science.

Harary gained his PhD from the University of California in 1948, and then moved to the University of Michigan in Ann Arbor, rising to Professor of Mathematics. On retirement from Michigan, he was appointed Professor of Computer Science at the New Mexico State University in Las Cruces.

Harary wrote and co-authored over 700 papers, as well as publishing eight books, including his celebrated and oft-cited book *Graph Theory* [17] in 1969. This book defined, developed and directed the path of modern graph theory and was widely used in university undergraduate courses. He was one of the founders of the *Journal of Combinatorial Theory* in 1966 and the *Journal of Graph Theory* in 1977.

Among his most significant research included work on graph enumeration, graph Ramsey theory and signed graphs. Graph enumeration is the counting (up to isomorphism) of graphs of a specified kind and Harary was an expert. His book, co-authored with Edgar M Palmer, entitled *Graphical Enumeration* [18] covered most of his work on the first of these subjects. Ramsey theory is named after a British mathematician and philosopher Frank P Ramsey (1903–1930) and is a branch of mathematics which investigates how many elements of some structure there must be to guarantee that a particular property must hold. Harary published a number of papers on this topic. His work in the third subject is indicated from the papers entitled *On the notion of balance of a signed graph* [19] in 1953–54, and *On local balance and N -balance in signed graphs* [20] in 1955, from which emerged a new

branch of graph theory, signed graphs, which grew out of a problem of theoretical social psychology.

9.4 Gerhard Ringel (1919–2008) & John William Theodore Youngs (1910–1970)

Gerhard Ringel, was born in Kollnbrunn in Austria, was a German citizen who became a naturalised American citizen. He became a professor of mathematics at the University of California in Santa Cruz in 1970 after a successful academic career in Germany, lecturing in Bonn, Frankfurt, and at the University of Berlin, where he was professor of mathematics and director of the Mathematics Institute. He published many papers and books on combinatorics and graph theory. In 1952, he published a paper [21] which addressed the colouring of maps on non-orientable surfaces and proved the Heawood conjecture for non-orientable genus $p > 1$ — the only exception was Klein bottle. As stated by G. A. Dirac in his review of the paper [22]:

This paper makes a very considerable contribution towards the solution of this problem for one-sided surfaces.

In 1959, Ringel published a significant book on the colouring of graphs, entitled *Färbungsprobleme auf Flächen und Graphen* [23]. This book and his numerous papers were considered major contributions to topological graph theory. Several of these papers were written jointly with Youngs who was at the University of California when Ringel arrived there at Youngs' invitation for the academic year 1967–68.

Ted Youngs was an American, born in Bilaspur, India, but educated in the USA, gaining his doctorate in 1934 from Ohio State University. He taught at Ohio State, Purdue and Indiana University where he remained for 18 years, served in the US Air Force during World War II, and was a consultant to industry and the Institute of Defence Analysis. In 1964 he moved to the University of California at Santa Cruz, where he remained until his

death. His numerous papers on topology contained many significant results, with his best early work concerning the abstract concept of a surface.

Between 1963 and 1968 Ringel and Youngs, published many joint papers on the Heawood conjecture; they investigated the minimum number of colours required to colour any map or graph on a given surface — this is the Heawood number. As Ringel had completely solved the problem for non-orientable surfaces, they concentrated on orientable (or two-sided) surfaces. Their proof required the solving of 12 separate subcases which arose from the denominator 12 that appeared in work of Heffter in 1891 (see Chapter 3). They (and others) had dealt with several cases earlier, but the final three cases were resolved during Ringel's sabbatical in California in 1967–68. Their work resulted in the paper *Solution of the Heawood map-coloring problem* [24], published in 1968, which brought together all of their findings providing an answer for every surface except the plane and the subsequent book *Map color theorem* [25], written in 1974 after Youngs death.

9.5 Oystein Ore (1899–1968) & Joel Stemple (b. 1942)

Yet another collaborative pair were Oystein Ore and his research assistant Joel G Stemple. Ore was born in Norway and was later to receive the Knight Order of St Olav, as Veblen had done 18 years earlier.

In a one page note published in 1960, *Note on Hamiltonian circuits* [26], Ore added to further conditions such that a graph is Hamiltonian and proved that:

If G is a simple graph with n (≥ 3) vertices, and if the sum of the valences of each pair of non-adjacent vertices is at least n , then G is Hamiltonian.

Ore presented the American Mathematical Society Colloquium Lectures in 1941, which resulted in his book *Theory of Graphs* [27]. In 1963 he published a book *Graphs and Their Uses* [28] which was written for high-school pupils, and in 1967, a book entitled *The Four-Color Problem* [29], which was variously described as a 'first rate complete

presentation' [30]; a 'classic book on the subject' [31]; and as 'authoritative and influential' and 'the first major book devoted exclusively to map colouring' [16].

In 1927 he settled in the USA where, in the mid-1960s, Stemple, who wished to research in graph theory, joined him at Yale. Together in 1968, they wrote a paper entitled *Numerical calculations on the four-color problem* [32] that extended the Birkhoff number to 41.

9.6 Frank Bernhart (b. 1945)

Another American mathematician who went into the family business was Frank Bernhart, son of Arthur (see Chapter 8). Frank Bernhart was on the staff of the University of Waterloo at the same time as Tutte. During the 1970s, he produced some significant work on the concept of reducibility. He also gained a reputation for being an expert in uncovering errors in purported 'proofs' of the four-colour theorem. His 1975 paper, *The four color theorem proved by multi-linear algebra???* [sic] [33] demonstrated the errors in two published 'proofs' of the four-colour theorem.

In a paper [34], published in 1971, he proposed a combinatorial condition for a planar graph that he then used to develop a new proof of Kuratowski's theorem on planar graphs. Another paper, *A three-five color theorem* [35], was published in 1973, and had three principal results. The first was, that if a plane graph G has a given face f , then G has a 5-colouring that gives a 3-colouring of the boundary of f . The second and third followed from the first: if G becomes planar upon the removal of one edge, then G can be 5-colourable, and if H is a connected induced subgraph of G , then any 2-colouring of H can be extended to a 5-colouring of G .

9.7 Kenneth Appel (b. 1932) & Wolfgang Haken (b. 1928)

They say that all good things come to an end, and that was certainly true for the quest for a solution to the four-colour theorem. In 1976, Kenneth Appel and Wolfgang Haken, two colleagues at the University of Illinois in Urbana–Champaign, provided a proof. Controversially, the proof was based on massive amounts of computer time that was needed to check an original list of 1936 reducible configurations (which by the time of publication they were able to cut to 1482). In mid-1974, John Koch was a graduate student in the computer science department at the University of Illinois when Appel and Haken were looking for someone to help in the necessary computer programming. Accepting the challenge, Koch created programmes for testing the reducibility of, initially, configurations of ring-size 11 and then ring-sizes 12, 13 and 14. Ring-size is clearly defined in [16] as ‘The number of countries surrounding a configuration. If there are k surrounding countries, the configuration has ring-size k ’.

Some mathematicians were slow to accept the proof as, at that time, most proofs were relatively straightforward to follow and had an associated hint of elegance. In their 1977 book, *The Four Color Problem* [31], Thomas Saaty (b. 1926) and Paul Chester Kainen (b. 1943) summed up the feeling of a large segment of academia, saying:

The sophisticated technique of Haken and Appel appears to have succeeded in proving that the 4CC is true. We say “appears to have succeeded” since their proof involves the computed-facilitated analysis of 1936 special cases, and will thus require several years for thorough checking. Even then, there will probably persist some lingering doubt among many scientists because of the elaborateness of the argument.

Their proof was the development of the works of earlier mathematicians (particularly Birkhoff) and the thorough investigation of unavoidable sets of reducible configurations of ring size 6 by Arthur Bernhart in his 1947 paper (see Chapter 8).

It was probably due to Frank Bernhart's reputation as an exposé of false claims of 'proofs' of the four-colour problem that prompted the editors of *Mathematical Reviews* to ask him to evaluate Appel and Haken's papers: *Every planar map is four colorable. Part I: Discharging* [36] and (with J Koch) *Every planar map is four colorable. Part II: Reducibility* [37]. His review mentions 'The authors of this proof' [38], but does not categorically state that the reviewer accepts that it is a proof of the four-colour theorem. The situation is best summed up by Robin J Wilson in his book *Four Colours Suffice* [16], where he writes:

The eventual solution, by Wolfgang Haken and Kenneth Appel in 1976, required over a thousand hours of computer time, and was greeted with enthusiasm but also with dismay. In particular, mathematicians continue to argue about whether a problem can be considered solved if its solution cannot be checked directly by hand.

Tutte's comments on Appel and Haken's proof were included earlier in this chapter, and these gave credibility and assistance to the acceptance of the proof within the mathematics community.

Two referees were selected to review the papers, one for the discharging part and one to examine the reducibilities, and after a number of small discrepancies were resolved, the proof was accepted. In 1994 another group of mathematicians, Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas, produced a shorter and more systematic proof, based on considerably fewer (but still over 600) reducible configurations and using the same arguments as those of Appel and Haken. Their work [39] confirmed the rightful acceptance of the 1976 proof.

In conclusion

The development of graph theory in America, from the employment of James Joseph Sylvester at The Johns Hopkins University in 1876, through the contributions of Charles Sanders Peirce, Oswald Veblen, George David Birkhoff, Philip Franklin and Hassler

Whitney, to the time of Bill Tutte and beyond was truly remarkable. Their contributions were of considerable significance to graph theory and to the eventual solution to the four-colour problem. They were not the only mathematicians of the USA to advance the subject, but were undoubtedly the most significant.

Although their work as graph theorists has been of primary importance here, it must also be remembered that many of them made outstanding contributions to other areas outside graph theory. This included support of their country during the course of two world wars, achievements in the administration of institutions of higher education, and in some cases, the valuable assistance in rescuing and deploying German-speaking refugees. However, above all of these achievements was the almost unanimous enthusiastic dedication to the development of young scholars, to the benefit of the USA, and to the advantage of all.

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Appendix I

The early years of graph theory

Graph theory can be said to have begun with Leonhard Euler in 1736. His proof that the Königsberg bridges problem has no solution is widely considered as the first step in the story of what was to become graph theory, although he did not use any graphs in his solution. Over the next 160 years the subject developed, with significant contributions by Gustav Robert Kirchhoff, famous for his laws on electrical circuits, who introduced *spanning trees* and *fundamental cycles*; Thomas Penyngton Kirkman, who worked on polyhedra; and William Rowan Hamilton, after whom *Hamiltonian cycles* are named; and Arthur Cayley, who enumerated *trees*, and his close collaborator James Joseph Sylvester. Also during these early years, the most celebrated problem in graph theory was born — the *four-colour problem*; the first known written mention was by Augustus De Morgan. Alfred Kempe and Cayley were among those who took up the challenge to provide a solution to this simple to state, but a difficult to solve problem.

Further information on graph theory can be found in [1], [2], [3] and on map colouring in [4], [5], [6], [7]. The book [8] gives a comprehensive history of Arthur Cayley and his mathematics as does [9] on the life and work of T P Kirkman.

1.1 The eighteenth century

The origins of graph theory can be traced back to an article written in 1736, although it would not be until the 1930s that the term *graph theory* would begin to be applied to a branch of mathematics. Leonhard Euler (1707–1783) was one of the leading mathematicians of the eighteenth century. His prodigious output included contributions to many areas of mathematics.

Euler has been called the founding father of *topology*. His 1736 paper [10], although important to graph theory as its origin, is considered, perhaps, one of his less significant papers to mathematics as a whole. This may be due partly to the problem that prompted his paper — *the problem of the Königsberg bridges*.

Königsberg was a city in Eastern Prussia built on the River Pregel, which divides into two branches within the city that encircle the island of Kneiphof². The city is served by seven bridges spanning the Pregel and, as Euler stated in his paper [10]:

Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he could cross each bridge once and only once. I was told that some people asserted that this was impossible, while others were in doubt; but nobody would actually assert that it could be done.

Euler proved that such a route is not possible.



Königsberg reproduced from a map published in the seventeenth century. [11]

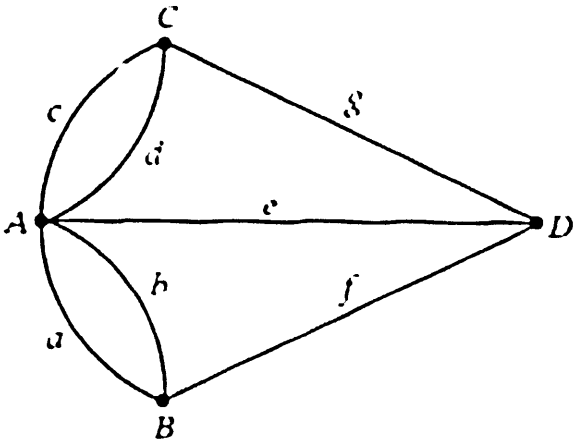
The paper included a detailed argument providing a general proof for whether such a route is possible or not for any number of areas of land and by any number of bridges joining those areas. Euler provided a set of rules stating ‘With these rules, the given problem can always be solved.’

Euler applied two significant ideas to the problem. The first was that he substituted a topological equivalent illustration, but eliminating metrical ideas, in place of the map of

² Today the city of Königsberg is known as Kaliningrad and the river Pregel as Pregolya.

Königsberg incorporating letter designations for those functions to be used in the evaluation of the problem; and his second was to define the problem from the illustration thereby making it redundant. He proved the impossibility of the question by counting the number of bridges emerging from each area.

The following diagram represents the connections in the Königsberg bridge problem. Euler did not draw such a graph as this type of illustration was first drawn in 1892 by W. Rouse Ball (1850–1925) [12].



This diagram is an example of what is known as a *graph*. A graph is constructed from a set of *vertices* (A, B, C , etc.), a set of *edges* (a, b, c , etc.), together with a list of which edges join which pairs of vertices. A closed path which traces out each edge of a graph once and only once returning to its starting point is now called an *Eulerian path*.

Drawing graphs in this way are now called diagram-tracing puzzles. These were addressed by other scholars, including the Frenchman Louis Poincot (1777–1859) in 1809 and the German Johann Benedict Listing in 1847.

Euler left several more legacies for future combinatorialists. Two of these were *Euler's formula* for polyhedra and his researches into partitions of integers. His polyhedron formula was included in a letter [13] from him to Christian Goldbach (1690–1764) in November 1750. It related the numbers of V vertices, E edges and F faces of a polyhedron and is often called *Euler's polyhedral formula*: it can be written

$$V + F = E + 2.$$

The letter detailed Euler's thoughts regarding the properties of solids bounded by plane faces, and provided the necessary algebra for analysing the subject, and in it Euler confessed that he had been unable to prove the formula. It was not until 1794 that Adrien-Marie Legendre (1752–1830) provided a satisfactory proof, albeit a metrical, rather than a topological, one.

In the early eighteenth century, two French mathematicians, Pierre Raymond de Montmort and Abraham de Moivre, published combinatorial works specifically on the *principle of inclusion and exclusion*, also known as the *sieve principle*, which deals with the evaluation of finite sets. The formula for $n = 2$ is:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

This principle has a number of uses in combinatorics including the counting of all derangements of a finite set. (A *derangement* of a set is a bijection from a set A into itself that has no fixed points). The inclusion-exclusion property was generalised by Whitney (see Chapter 6).

Also in this century, Alexandre-Théophile Vandermonde (1735–1796) published a paper *Remarques sur les problèmes de situation* [14]. This work was primarily devoted to the chessboard problem known as the knight's tour — that is, to determine a sequence of knight's moves so that a knight visits each square on a chessboard once and once only, returning to the square from which it commenced the tour — a problem that Euler had worked on earlier. This indicates how diversified are the problems that are now encompassed within graph theory.

1.2 The nineteenth century

There was little advance in graph theory after Euler until the early years of the nineteenth century. However, in the 1810s, two notable contributions to the subject were published. The first, a paper entitled *Démonstration immédiate d'un théorème fondamental d'Euler*

sur les polyèdres, et exceptions dont ce théorème est susceptible [15] by Simon-Antoine-Jean Lhuilier (1750–1840), a Swiss mathematician, was associated with Euler’s polyhedral formula and was published in 1811. The second was by Augustin-Louis Cauchy (1789–1857), who became the leading French mathematician of the first half of the nineteenth century, and was accorded international acclaim for his seminal work in analysis and group theory. In 1813, his paper *Recherches sur les polyèdres-premier mémoire* [16] investigated Euler’s polyhedra formula. Initially he projected the polyhedra onto a plane surface, giving what is now called a plane graph, and as he excluded the external region of the projection his version of Euler’s formula became $n - m + f = 1$ (where n , m and f denote, respectively, the numbers of vertices, edges and faces). The paper included a general proof of his development of Euler’s formula that was applicable to graph theory. In its simplest form it is:

Let G be a connected planar graph, and let n , m , and f denote, respectively, the numbers of vertices, edges, and faces in a plane drawing of G . Then $n - m + f = 2$.

As is seen in the thesis, these ideas were relevant to the work of many of the mathematicians mentioned, including Kempe, Heawood, Veblen, Franklin and Whitney.

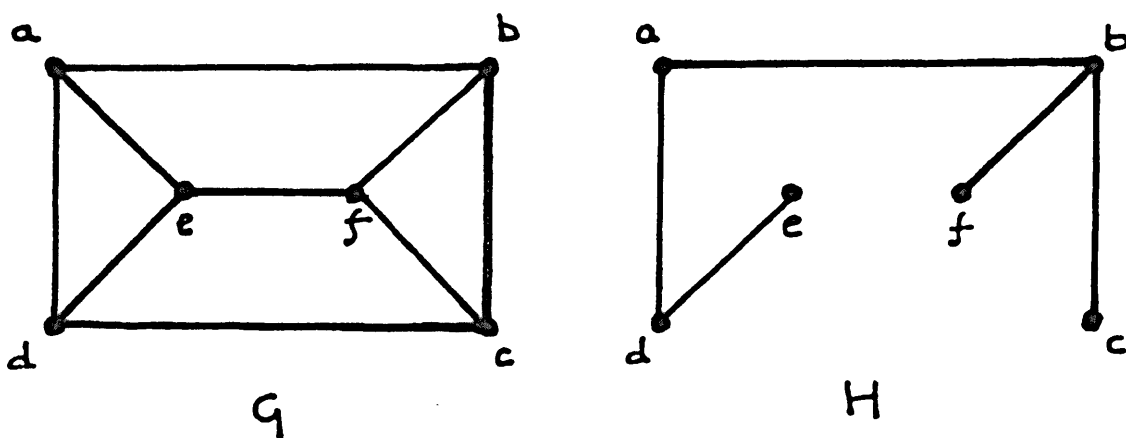
Gustav Robert Kirchhoff (1824–1887)

It was not until the middle of the century that further significant advances in graph theory were made. Kirchhoff is mostly remembered for his famous laws on the flow of currents within electrical circuits, which he formulated at the age of 21 whilst attending the University of Königsberg. These laws, now known as *Kirchhoff’s laws*, are used to develop equations, from which electrical circuits can be analysed and the values of currents and potentials calculated.

The laws were published in two papers, in 1845 [17] and 1847 [18]. The first paper developed the equations, while the latter (a major contribution to the theory of graphs), provided a method for determining how many equations are necessary to yield a solution.

From the graph theory point the wires of an electrical circuit may be considered as the edges of a graph and the terminals connecting the wires as vertices. His *Voltage law* could then be applied to every cycle of edges and his *Current law* to each vertex. This generated a set of linear equations, not necessarily independent of one another. In his 1847 paper Kirchhoff explained how a fundamental set of cycles may be developed and proved that, for any connected graph with m vertices and n edges, a fundamental set always contains $n - m + 1$ cycles.

Kirchhoff's two papers included the development of what would be called a *spanning tree* (a subgraph H of a connected graph G that includes all the vertices of G and is also a tree, as shown in the diagram below). These ideas were later taken up by Oswald Veblen (see Chapter 4).



At the time Kirchhoff's work was not recognised as significant to what was to become graph theory, but he was the first to use algebraic methods when studying electrical networks. Over the next century several leading mathematicians used his ideas and techniques in the development of both topology and graph theory.

Johann Benedict Listing (1808–1882)

In 1847 Listing published a significant paper in what was to become topology. Topology is an area of mathematics concerned with properties that are preserved under continuous deformation of objects; for example deformations that include stretching but not tearing or

gluing. The paper entitled *Vorstudien zur topologie* (Introductory studies in topology) [19] studied geometry in terms of position rather than angles and distances and included work on diagram-tracing problems. Although he had used the word *topology* in correspondence for some years, possibly as early as 1836, it was in this paper that the word was first published.

He continued his work on topology with a paper in 1861 [20]. In it he defined *complexes* and investigated their connection with Euler's formula. He used the word 'complexes' as they were assembled from simpler objects and examined their topological properties. This work was described in [2]:

... especially the question of how such properties affect the generalizations of Euler's formula. The properties were given names such as 'periphaxis' and 'cyclosis', borrowed from the biological sciences.

This work was developed by other mathematicians including Henri Poincaré (1854–1912) who applied algebraic methods to the subject. This paper also included an example of a one-sided surface which was also discovered by August Ferdinand Möbius (1790–1868), independently a few months later, and is now known as a *Möbius band*.

In his development of algebraic topology, Poincaré defined cells as the building blocks of Listing's complexes and applied Kirchhoff's approach of representing equations in matrix form (called the *incidence matrix* of the graph), in place of a set of linear equations which described how the cells were used to construct complexes. The cells included 0-cells (vertices) and 1-cells (edges) and these were used to draw a graph. These ideas would also be taken up by Oswald Veblen (see Chapter 4).

Thomas Penyngton Kirkman (1806–1895)

Thomas Kirkman was a Church of England vicar and mathematical amateur (although he had a degree that included mathematics) of the kind that was regularly found in Britain at

the time. He published over 60 substantial mathematical papers and many more minor ones.

One of his areas of study was what he called ‘polyedra’, and in 1855, he published the first [21] of a series of papers on the subject — a topic in which he retained an interest for the rest of his life. In 1855, he presented a paper [22] to the Royal Society that considered the question:

Given the graph of a polyedron, can one always find a circuit which passes through each vertex once and only once?

The answer to this question was no. Whilst in many cases such a circuit can be found, there were others for which a circuit does not exist. However, his effort was not wasted, as in another part of the paper he gave conditions for a class of graphs that have no complete cycle of the kind required. He included a general proof of the fact that a polyhedron with an odd number of vertices and each face having an even number of edges, has no circuit that passes through all of the vertices.

William Rowan Hamilton (1805–1865)

William Hamilton was the leading mathematician and astronomer in Ireland during the 19th century. In 1835, he developed equations of motion for a general dynamical system, which incorporated the symbol H , now known as the *Hamiltonian function*; H is defined as a function of time and the velocities and positions of the constituent masses of a system.

For some centuries complex numbers had been viewed with distrust and Hamilton spent fifteen years studying them. He believed that the expression $a + ib$ was not the correct way to define a complex number, rather it should be regarded as an ordered pair of real numbers, (a, b) , under certain rules. He tried to extend his idea to complex number triples in three dimensions (a, b, c) , written $a + ib + jc$, where i and j are two distinct and independent square roots of -1 . He developed ‘quaternions’, a number quadruple of the form $a + ib + jc + kd$ where $i^2 = j^2 = k^2 = ijk = -1$ although the multiplication rule is not

commutative as the order in which two quaternions are multiplied results in different answers. a is real number called the scalar part and $ib + jc + kd$, the vector part. The vector part is the expression of the line from the origin in three-dimensional space. Hamilton was obsessed with this work, which was less useful at the time than he had hoped, but his discovery of non-commutative systems led to a revolution in the development of algebra. [23]

There are many systems of non-commutative algebra and one of these, discovered by Hamilton and named *The Icosian Calculus* [24], can be interpreted in terms of cyclic paths on the graph of a regular dodecahedron. This he developed into a game called *The Icosian Game*, which comprised a number of problems associated with tracing out paths and cycles on a dodecahedral graph. A second game was described in [25] thus:

There was another version of Hamilton's game, involving a solid dodecahedron ... and known as 'The Traveller's Dodecahedron' or 'A Voyage Round the World'. In this game, the vertices represented twenty important places: Brussels, Canton, Delhi, and so on, ending with Zanzibar. Each vertex was marked by a peg, and a thread could be looped around these pegs to indicate a path, or circuit. A complete circuit, passing through each place once only, was called a 'voyage round the world'.

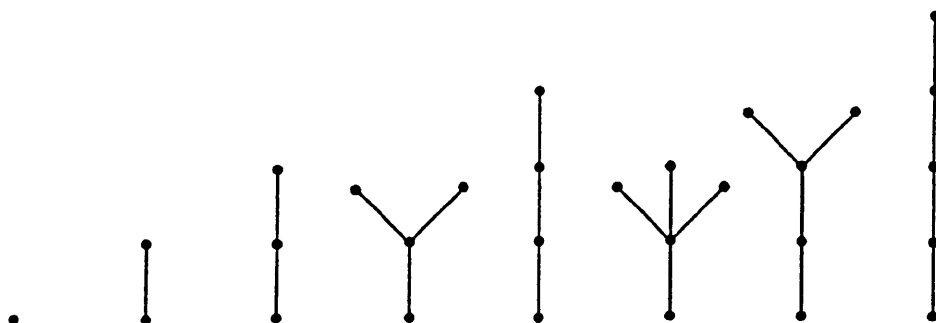
A circuit which passes through each vertex of a graph is now known as a *Hamiltonian cycle* and the corresponding graph is said to be *Hamiltonian*. Hamilton added to this work in his paper of 1856 [26].

It should be recorded that Kirkman, who independently developed more general and earlier work on paths and cycles and was the first to publish his ideas, did not receive the historical credit he deserves. As has happened many times, someone else, this time Hamilton, has had his name associated with terms that are used as standard today. It is known from letters between them that Hamilton visited Kirkman at his rectory during August 1861 and they enjoyed a mutual respect, albeit brief, as Hamilton died in 1865 [25].

Arthur Cayley (1805–1865)

Arthur Cayley was an English mathematician and barrister who made major contributions to many areas of mathematics. It was during his legal training that he met James Joseph Sylvester, who features prominently in Chapter 2 of this thesis, and they became lifelong friends and collaborators on mathematical matters, including invariant theory and what was to become graph theory.

For his part, Cayley produced a number of graph theory papers between 1857 and 1889. In 1857, he published the first paper [26] to use the word *tree*, as is now used in graph theory, a tree is defined as a connected graph that contains no cycles, although both Kirchhoff (spanning tree) and Karl Georg Christian von Staudt (1798–1867) had used the idea around ten years earlier; as a consequence, the number of edges is one fewer than the number of vertices, and a connected graph with this property must be a tree. The study of trees stemmed from the study of operators in the differential calculus.



The trees with up to five vertices.

His paper, inspired by Sylvester's work on what he called 'differential transformation and the reversion of serieses', dealt with rooted trees only. The first part of the paper was given over to explaining the correspondence with differential operators then progressed to the problem of calculating the number A_n of rooted trees with n edges.

As described in [2] 'he solved this problem by using two elementary techniques and one clever idea'. The first technique was to replace a sequence of numbers, A_0, A_1, A_2, \dots by a generating function, $A(x) = A_0 + A_1x + A_2x^2 + \dots$ then developing the argument with the

function in place of the sequence. The second technique was to remove the root and the edges meeting it from the rooted tree, resulting in a collection of new rooted trees whose roots are the vertices adjacent to the original root. He then developed a generating function for the number of rooted trees:

$$1 + x^{r+1} + x^{2(r+1)} + \dots = (1 - x^{r+1})^{-1}$$

where r is the number of edges of one particular rooted tree. Multiplying together all the generating functions for rooted trees gives:

$$(1-x)^{-1} (1-x^2)^{-A_1} (1-x^3)^{-A_2} \dots$$

His ‘clever idea’ was that he could see that the coefficients of the powers of x in the product are developed exactly as those of the generating function for the numbers of rooted trees. This means that the above product is equal to:

$$1 + A_1x + A_2x^2 + A_3x^3 + \dots$$

The paper concluded with mention of a method of calculating B_r , the number of trees with r free branches. In 1881 he published a short paper [27] which developed a method for counting unrooted trees.

Both Cayley and Sylvester also studied chemical molecules. In 1874, Cayley presented a paper *On the mathematical theory of isomers* [28]. This short paper allowed Cayley to apply his work on trees to another field of study. Isomers are compounds that have the same chemical formula, but different atomic configurations. This was one of a number of papers where he applied trees to his work on chemical compositions. Two further papers, in 1875 [29] and 1877 [30], also dealt with the connection between trees and chemical composition. These included a tree-counting method for finding the number of compounds C_nH_{2n+1} (alkanes or paraffins) with a prescribed number of carbon atoms. This area of his scholarship was described in [2] as:

This fusion of mathematical and chemical ideas inspired some of the terminology which is now standard in graph theory, including the word ‘graph’ itself.

Over a period of thirty years Cayley occasionally returned to his study of trees, and in 1889 he published a paper [31] outlining what is now known as *Cayley's theorem*. The problem posed in the paper was: given n labelled vertices, how many ways t_n are there of joining the vertices to form a tree? The theorem included the formula $t_n = n^{n-2}$ which was used to calculate t_n — for example, $t_4 = 16$. The ‘proof’ he provided was deficient, as he considered the case $n = 6$ only and the argument he offered could not easily be applied to larger values of n . However, since then there have been a number of proofs; perhaps the best was by the German mathematician Heinz Prüfer (1896–1934) in 1918.

James Joseph Sylvester (1814–1897)

Sylvester was another English mathematician who became a barrister. He was also an actuary for eleven years. As mentioned earlier, Sylvester met Cayley during their legal training, discovered their mutual passion for combinatorial mathematics in particular. Despite their very different personalities they became lifelong friends, and during the 1850s worked together on the algebraic theory of invariants. Sylvester worked with Kempe on linkages, making significant discoveries. Later he worked on number theory, invariant theory and published a proof of ‘Newton’s rule’ for the roots of equations. Sylvester’s contribution to graph theory in America is explored in Chapter 2.

1.3 The four-colour problem

The four-colour problem asks — can every map drawn on the plane be coloured with at most four colours so that no two neighbouring countries are coloured the same?

Augustus De Morgan (1806–1871)

Augustus De Morgan was a graduate of Trinity College, Cambridge, who became professor of mathematics at University College, London. He was an eccentric dogmatic

man whose place in the story of graph theory is that the first written mention of the *four-colour conjecture* was in a letter from him to Hamilton, dated 23 October 1852; they corresponded for thirty years. The letter was written the same day that one of De Morgan's students, Frederick Guthrie (1833–1886), had asked him about a map colouring problem that his brother Francis Guthrie (1831–1899) claimed to have proved. Francis Guthrie's 'proof' did not stand up to interrogation as in [32] his brother Frederick wrote:

Some thirty years ago, when I was attending Professor De Morgan's classes, my brother Francis Guthrie, ... showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity of colour in lineally contiguous districts is four. I should not be justified, after this lapse of time, in trying to give his proof, ...

The quest for a solution to the four-colour problem had begun and a proof was published, a century and a quarter later.

Hamilton did not appear to be very interested in the topic as his reply asserted 'I am not likely to attempt your quaternion of colours very soon' [33]. Perhaps because Hamilton seemed reluctant to pursue the colouring problem, De Morgan approached other mathematical friends hoping they would take up the challenge. One of these was William Whewell, Master of Trinity College, Cambridge. In 1860, De Morgan reviewed a book by Whewell, *The Philosophy of Discovery, Chapters Historical and Critical*, in the literary journal *The Athenaeum* and his review included a description of the four-colour problem [34]. The review, rediscovered in 1976, was then believed to be the first publication of the problem.

It is now possible to revise the date of the earliest known publication. This is due to a paper [35] by Brendan D McKay, in which he cites a letter that appeared in the *Miscellanea* section of *The Athenaeum* on 10 June 1854. The letter read:

Tinting Maps. — In tinting maps, it is desirable for the sake of distinctness to use as few colours as possible, and at the same time no two conterminous divisions ought to be tinted the same. Now, I have found by experience that *four* colours are necessary and sufficient for this purpose, — but I cannot prove that this is the case, unless the whole number of divisions does not exceed five. I

should like to see (or know where I can find) a general proof of this apparently simple proposition, which I am surprised never to have met with in any mathematical work. F.G.

Professor McKay comments that:

The magazine does not identify “F.G.”, but the short period of time between this letter and the known interaction between Francis and Frederick Guthrie makes it highly likely that one of them was responsible. It doesn’t seem possible to identify which of the brothers it was, but I favour Francis for the following, inconclusive, reason. In 1880, Frederick carefully attributed the discovery to Francis and did not mention having studied the problem himself [5].

The reference [5] above is reference [32] in this Appendix.

In his earlier letter to Hamilton, De Morgan asserted that if a map has four countries, each neighbouring the other three, then one of the countries must be completely surrounded by the remaining three. In his letter to Whewell De Morgan stated that this was an axiom that had lain ‘wholly dormant’ until it was connected to map colouring problems.

After De Morgan’s death, Cayley also worked on the four-colour problem. On 13 June 1878, at a meeting of the London Mathematical Society, he raised a query that was recorded in the Society’s *Proceedings* [36] as:

Questions were asked by Prof. Cayley F.R.S. – Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four colours are required, so that no two adjacent counties should be painted in the same colour.

This was repeated in a report of the meeting in *Nature* on 11 July 1878 [37]. [These reports were, for many years, believed to be the earliest printed references to the four-colour problem].

In 1879 Cayley published a short note [38] in the *Proceedings of the Royal Geographical Society* setting out to describe succinctly the difficulties inherent in tackling the four-colour problem. He suggested that it might be feasible to develop maps that required disproportionately large numbers of colours to ensure that no two regions sharing a common boundary be the same colour. The paper included a positive suggestion that when developing a proof for the four-colour conjecture, restrictions could be imposed on

maps, a portent of things to come. One restriction was that they could be *cubic maps* (those with exactly three countries at each meeting point), as this is no less general than the full problem. He also pointed out that if the four-colour problem were true, then a colouring could be found so that only three colours are adjacent to the exterior boundary.

Alfred Bray Kempe (1849–1922)

Kempe, a former student of Cayley, whose early mathematical work was associated with the application of geometry to mechanical linkages, particularly with mechanisms that trace out a straight line, such as the movement of a piston in a steam engine. His interest in the subject had been triggered by a lecture given by Sylvester at the Royal Institution in 1874 [39]. Kempe's work culminated in his presentation of a series of lectures at the Royal Institution in London in 1877, entitled *How to draw a straight line: A lecture on linkages* [40]. The publisher Macmillan collated the lectures and published a 51-page book, which became a standard for the topic. It was principally for his work on linkages that he was made a Fellow of the Royal Society.

He is, however, remembered most for his celebrated (but fallacious) proof of the four-colour conjecture. His interest in the topic had been aroused by Cayley's query to the London Mathematical Society [36], which Kempe attended, and Cayley's memoir [38] in the *Proceedings of the Royal Geographical Society* in April 1879. Shortly afterwards, on 17 July 1879, Kempe published a preview of his 'solution' in *Nature* [41]. Shortly after Kempe published 'simplified' versions, one an untitled abstract [42] in the *Proceedings of the London Mathematical Society*, and the second in *Nature*, under the title *How to colour a map with four colours* [43].

Kempe's original full paper, *On the geographical problem of the four colours* [44], was published in the second volume of the *American Journal of Mathematics* in 1879, having been requested by Sylvester, and it is in this work that he claims to have proved the four-

colour conjecture. The paper and its impact on graph theory in America are reviewed in Chapter 2. Unfortunately, it contained a fatal error, which was uncovered eleven years later, during which time his proof had been generally accepted. It was Percy John Heawood (see Chapter 3), who had heard of the four-colour problem from Henry Smith, Savilian Professor of Geometry at Oxford University, who found the error in 1890. Indeed, Kempe and other mathematicians, including Peter Guthrie Tait (see Chapter 3) and William Edward Story (see Chapter 2), published further papers containing so-called ‘improved’ versions; however all these works included the fundamental error contained in the original paper.

On 9 April 1891, at a meeting of the London Mathematical Society, Kempe admitted his error and recognised the work Heawood had done to uncover it. However he stressed that Heawood’s ‘criticism applied to my proof only and not to the theorem itself’. As other scholars published supposed versions of Kempe’s ‘proof’ during the decade after the paper was published, all of them containing the false argument, it can be assumed that at the time his paper was widely accepted as including a valid proof.

To remember Kempe solely because of this error is to do him an injustice, as his paper provided the foundations upon which future mathematicians would base their work; in fact, some hundred years later, the computer-aided proof published in 1976 made use of two of his ideas.

The first was the concept of *unavoidability* — that is, that it is impossible to construct a map without at least one of four specified (unavoidable) configurations: a region with two neighbours, one with three neighbours, one with four neighbours, and one with five neighbours. The second was the concept of *reducibility*. These two concepts formed the basis on which Kenneth Appel and Wolfgang Haken developed their eventual proof in 1976 (see Chapter 9). Kempe’s short list of four unavoidable sets needed to be extended to 1936 distinct cases in the proof, which then showed that all these cases were reducible. Kempe

also remarked that colouring problems of maps may be expressed in terms of vertices and edges of plane graphs only. In this he made use of the concept of duality although he did not develop the matter further.

Conclusion

Other combinatorial papers published during the early years of graph theory included work on diagram-tracing puzzles, the number of ways of completing a game of dominoes using all twenty-eight pieces, and the mathematics involved with escaping from mazes or labyrinths. These are not included in this thesis as they did not figure in the work done by American mathematicians.

The chronology of what was to become graph theory has now come from Euler in 1736 towards the end of the nineteenth century. The work on graph theory done in Europe would influence mathematicians in the USA. Indeed, in the last quarter of that century, a number of notable scholars made significant, but different, contributions to the development of mathematics in America, and in particular to the development of graph theory.

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Appendix II

Biographies

Biographies of the following are included in this Appendix:

George David Birkhoff

Arthur Cayley

Augustus De Morgan

Philip Franklin

William Rowan Hamilton

Percy John Heawood

Lothar Wilhelm Julius Heffter

Alfred Bray Kempe

Gustav Robert Kirchhoff

Thomas Penyngton Kirkman

Daniel Clark Lewis, Jr.

Sanders MacLane

Eliakim Hastings Moore

Simon Newcomb

Benjamin Peirce

Charles Sanders Peirce

Julius Peter Christian Petersen

William Edward Story

James Joseph Sylvester

Peter Guthrie Tait

Heinrich Franz Friedrich Tietze

William Thomas Tutte

Oswald Veblen

Hassler Whitney

Birkhoff, George David

Born 21 March 1884: Overisel, Michigan, USA

Died 12 November 1944: Cambridge,
Massachusetts, USA



G D Birkhoff was the grandson of George Birkhoff who emigrated to the USA in 1870 from Holland and settled in Illinois. His mother was also of Dutch extraction. From the age of 9, he showed considerable aptitude for mathematics, and at age 15 provided a solution to a problem, which appeared in the *American Mathematical Monthly*. In his mid-teens he proved that every integer of the form $a^n - b^n$ (a and b integers, $n > 2$), with the exception of $63 = 2^6 - 1^6$, has a prime divisor p which does not divide $a^k - b^k$ for any proper divisor k of n ; this led eventually in 1904 to a jointly published paper entitled *On the integral divisors of $a^n - b^n$* which appeared in the *Annals of Mathematics*, his first publication, while he was still an undergraduate student at Harvard University.

Birkhoff entered the University of Chicago in 1902 for one year, and then studied at Harvard University from 1903 to 1905, being awarded an AB degree in 1905 and an AM in 1906. He returned to the University of Chicago in 1905 for postgraduate study under Eliakim Hastings Moore, which was rewarded with a PhD in 1907; the thesis was entitled *Asymptotic Properties of Certain Ordinary Differential Equations with Applications to Boundary Value and Expansion Problems*.

Birkhoff's main contribution to graph theory was his two publications of 1912 and 1913 the latter being a pioneering paper which would prove to be significant in the development of the solution to the four-colour problem. In addition to his work on map-colouring, he made contributions to many mathematical topics, including number theory, the Riemann-Hilbert problem, the three-body problem, the calculus of variations, relativity, dynamic systems and suitability, Poincaré's geometric theorem, linear differential equations and the theory of difference

equations, but his main work was on dynamics and ergodic theory. He was awarded the *Quirini Stampia Prize* by the Royal Venice Institute of Science, Letters and Arts in 1918 and was the first recipient of the American Mathematical Society's *Bocher Memorial Prize* in 1923. In 1933, he was awarded an honorary doctorate from Harvard. The citation called him;

... first in our land among masters of mathematics, that great tool of science, greater still in the realm of pure imagination.

In total he received thirteen honorary degrees; the citation for his ScD from Brown University in 1923 included the phrase;

... youngest professor of mathematics at our oldest university, already recognized throughout America and Europe as a leading discoverer and interpreter in the most fundamental of sciences.

Others included ScDs from the University of Wisconsin in 1927 and the University of Pennsylvania in 1938. In 1933, on the occasion of the 500th anniversary of the founding of the university, the University of Poitiers awarded him a doctorate, as did the University of Paris in 1936 with a glowing tribute which recalled the role Birkhoff played in the creation of the Institute Henri Poincaré. He also received a doctorate from the University of Athens in 1937 and an LLD from the University of St. Andrews in 1938. The number of awards and honours bestowed on him by universities and learned societies around the world were an indication of Birkhoff's stature as a scholar.

He had a long association with the American Mathematical Society, being Vice-President in 1919, a Colloquium Lecturer in 1920, editor of the *Transactions* of the Society from 1921 to 1924, and President from 1925 to 1926. He also edited the *Annals of Mathematics* from 1911 to 1913 and the *American Journal of Mathematics* from 1943 to 1944. He was elected a member of numerous learned societies and invited to address them and international meetings. The lunar crater *Birkhoff* was named after him.

Birkhoff was, perhaps, not the most modern or diplomatic of people — in fact, in terms of viewing the younger generation of the time he was a throwback to older times when the senior male in a family came first and all other members of the household were required to support him or, if children, to do as they were commanded. A clue to this attitude was a comment that Birkhoff made to Vera Ames Widder shortly after her wedding in 1939, when he said, 'One career in a home

is enough' [1] — this to a woman who had received a PhD in mathematics from Bryn Mawr College; taught at Tufts University, Cambridge Junior College, the University of Massachusetts (Boston) and UCLA. Additionally, she had married David V Widder, who had been a postgraduate student of Birkhoff's at Harvard, and who after six years at Bryn Mawr College returned to Harvard and became a full professor.

Birkhoff has been described as 'by nature intensely social', but this was mainly demonstrated by his relationships with academic like-minded people of similar background. He was afforded many opportunities to develop his social skills and personality through the great many congresses and meetings he attended; his numerous visits to Europe; his extensive contacts with scholars around the world; and his numerous trips around the globe. He received many invitations to write and speak on a wide range of subjects and, in later years, was able to make significant contributions to the administrative side of scholarship.

For the last few years of his life he was aware that his heart was less than strong, but continued to work as hard as ever. He died in his sleep at the age of sixty years.

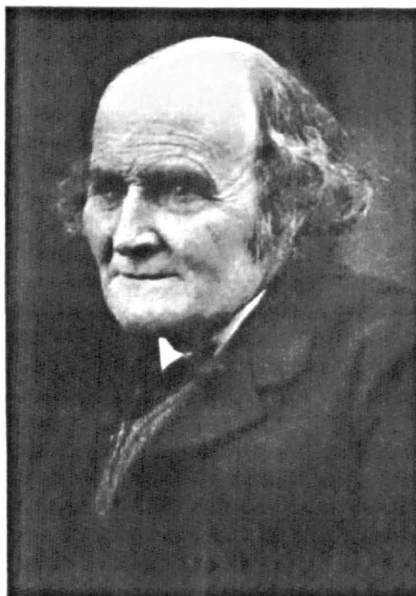
Rudolf and Gerda Fritsch wrote [2]:

Because of his creativity and versatility, Birkhoff had, as a teacher and as a researcher, a great impact on his numerous students. He was one of the most important American mathematicians at the beginning of the 20th century.

Cayley, Arthur

Born 16 August 1821: Richmond, Surrey, England

Died 26 January 1895: Cambridge, England



Arthur Cayley spent his young childhood in St. Petersburg, Russia, as his father worked there as a commercial agent. Cayley studied at Trinity College, Cambridge, showing significant ability, and graduated in 1842 as Senior Wrangler. In

October of that year, he was elected a fellow of Trinity, making him the youngest 19th-century scholar to achieve this at Cambridge. The rules required fellows to take holy orders within seven years of their appointment; this Cayley was not prepared to do and after a few years decided to leave Cambridge and undertake legal training in London. He enrolled in chambers in Lincoln's Inn and was called to the bar in 1849, spending 14 successful years working as a barrister during which time he produced nearly 300 mathematical papers. Then, in 1863, he was appointed to the newly created Sadleirian chair of pure mathematics at Cambridge (which had no attached religious requirements) a post he held until his death in 1895.

Cayley produced a number of graph theory papers between 1857 and 1889. In a paper of 1857 he used the word *tree*, as is now used in graph theory, for the first time and continued to publish, occasionally, work on trees for over thirty years. Like Sylvester he studied chemical molecules and was instrumental in developing the study of trees and their connection with chemical composition.

Cayley was one of the foremost mathematicians of the 19th century with contributions to many areas of mathematics including group theory, linear algebra and algebraic theory but his most important work is considered to be the development of algebra of matrices, non-Euclidian and n -dimensional geometry; in all he published over 900 mathematical papers.

De Morgan, Augustus

Born 27 June 1806: Madura, India

Died 18 March 1871: London, England

Augustus De Morgan attended Trinity College, Cambridge, graduating in 1827, and in the following year he was appointed professor of mathematics at University College, London. He was eccentric and dogmatic, and had strongly held



opinions that led him to resign on a matter of principle from his position at University College in 1831. He was reappointed in 1836, but in 1866 resigned for a second time.

De Morgan's major contributions to mathematics were in the fields of logic and set theory and he developed laws of duality that expressed the intersection of sets or joint statements of two propositions. These may be displayed symbolically as:

$$\overline{x \cdot y} = \overline{x} + \overline{y} \text{ and } \overline{x + y} = \overline{x} \cdot \overline{y} \text{ or } (A \cup B)' = A' \cap B' \text{ and } (A \cap B)' = A' \cup B'.$$

The laws may be expressed as saying; for each proposition comprising logical addition and multiplication, there is a dual proposition where the words addition and multiplication are interchanged. The American scholar Benjamin Peirce also worked on these laws independently, and Venn diagrams may be used to represent them symbolically.

Although De Morgan did not publish any graph theoretical work, he was instrumental in initiating the search for a solution to the four-colour problem. He urged Hamilton, amongst others, to study the subject, in a letter in 1852, which was believed to be the first written mention of the four-colour conjecture until 2012 (Appendix I).

De Morgan was dedicated to the history of mathematics and accumulated a large collection of books on the subject — particularly, volumes on arithmetic. He was an avid writer, producing many articles for the *Penny Cyclopaedia* between 1833 and 1843, as well as contributing on a regular basis to several journals and periodicals. On the death of his friend George Boole (1815–1864), he continued Boole's work on the calculus of propositions.

Franklin, Philip

Born 5 October 1898: New York City, USA

Died 27 January 1965: Belmont, Massachusetts, USA



Philip Franklin enrolled in the College of the City of New York in 1914. He received a BS in 1918, before going on to Princeton University for postgraduate study, receiving an MA in 1920 and a PhD in 1921 for his thesis *The Four Color Problem*. He remained at Princeton for a year as

an instructor of mathematics and then spent two years at Harvard University as the Benjamin Peirce Instructor. At this time Birkhoff, well ensconced at Harvard which had an anti-Semitic culture, and given that Franklin was Jewish as well as his connection to Wiener, his brother-in-law, it is not surprising that Franklin did not remain at Harvard [3].

In 1924, he moved to the Massachusetts Institute of Technology as an instructor in mathematics, becoming an assistant professor in 1925, an associate professor in 1930, and a full professor in 1937 — a position he held until his retirement in June 1964.

He was awarded a Guggenheim Fellowship for the year 1927–1928 and introduced topology and colouring problems to the Institute; the 1933–1934 issue of the *MIT Journal* included his discussion of the six-colour problem for the Klein bottle. This paper was also published in the *Journal of Mathematics and Physics* in 1934. He was secretary of the mathematics faculty for the five years before his retirement. He also served as chairman of the Institute's *Committee on Academic Performance* and his colleagues on the committee described him as an 'anonymous friend of students over many years.'

President Stratton of MIT also paid tribute to him, saying that Franklin was a teacher

... who has devoted himself with special distinction to the welfare of students and to the process of teaching.

Franklin published five graph theory papers, four on map-colouring and one which gave a shorter and alternative proof to that of Kirchhoff for the calculation of currents in electric circuits. In addition to his contributions to graph theory, he produced significant work on geometry, algebra, and analysis, including the calculus, differential equations, complex variables, and Fourier series. He was the author of some 60 research papers in the fields of geometry, topology, and analysis and wrote several exceptional books, three of which were used as texts for MIT courses. These books included *Differential Equations for Electrical Engineers* (1933), *Treatise on Advanced Calculus* (1940), *Methods of Advanced Calculus* (1944), *Fourier Methods* (1949), *Differential and Integral Calculus* (1953), *Functions of a Complex Variable* (1958) and *Compact Calculus* (1963) and the booklet *The Four Color Problem* in 1941.

In 1943, the College of the City of New York awarded Franklin the Townsend Harris Medal for

... the alumnus who achieved notable postgraduate distinction.

He was managing editor of the *Journal of Mathematics and Physics* from 1929 to 1945 and editor from then until his death; and served on the *Editorial Committee on Carus Monographs* and the *Chauvenet Prize Committee* of the Mathematical Association of America.

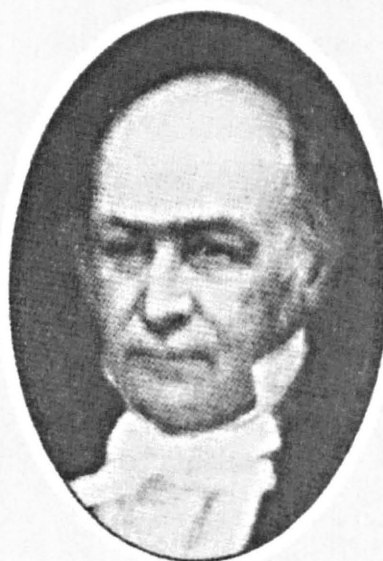
Philip Franklin died at the Massachusetts General Hospital in 1965. He has been described as an even-tempered and mild man with a friendly sense of humour. He was a kind, respected man who enjoyed considerable and variable levels of affection. At his memorial service, Dean Harrison of MIT described him as

...of almost Mr Chips proportions.

Hamilton, William Rowan

Born midnight 3/4 August 1805: Dublin, Ireland
(then part of the United Kingdom of Great
Britain and Ireland)

Died 2 September 1865: Dunsink, Ireland



William Rowan Hamilton's mother died when he was 12, and his father died two years later, but these events may have affected Hamilton less than most young boys experiencing the same tragedies, as he had been sent to live with an uncle at the age of 3. He showed signs of being exceptionally gifted at a very early age, and his uncle, the Reverend James Hamilton, a noted and accomplished linguist and polymath, could provide a more fitting environment for the young child's talents to develop. This proved successful as Hamilton was truly a child prodigy, showing startling ability for mathematics during his early years and studying many languages, modern, classical and oriental. He came to the attention of the academic world at the age of 17 when he discovered a significant error in Laplace's *Mécanique Céleste*. In 1823, he enrolled at Trinity College, Dublin, becoming an outstanding student in both the classics and sciences. He was appointed Astronomer Royal of Ireland, Director of the Dunsink Observatory, and Professor of Astronomy at Trinity College, before he had even graduated.

Hamilton made contributions to many areas of scholarship, including mathematical physics, dynamics and optics. He studied complex numbers, discovered *quaternions* and invented the *Icosian calculus* which was related to his work on quaternions. In graph theory he will be most remembered for having the Hamiltonian cycle named after him. He received international recognition when an optical prediction he had made was proved experimentally. He was knighted at age 30 and was considered Ireland's greatest man of science. He became President of the Royal Irish Academy in 1837, and whilst on his deathbed was informed that he had been elected the first foreign member of the National Academy of Sciences of the USA. His death was the result of an acute attack of gout.

Heawood, Percy John

Born 8 September 1861: Newport, Shropshire, England

Died 24 January 1955: Durham, England

Percy Heawood studied at Exeter College, Oxford, gaining degrees in both mathematics and classics, a BA in 1883 and an MA in 1887. He then became a lecturer at the Durham Colleges (later Durham University), becoming a professor in 1911 and retiring in 1939 at the age of 78. He was Vice-Chancellor for the period 1926 to 1928.



Heawood published many papers on map colouring, his first, in 1890, uncovered the error in Kempe's supposed proof, and his last was some 70 years later, in 1949. He was a scholar of ancient languages, including Latin, Greek and Hebrew. He sat on many committees, and held a number of official lay positions within the Church of England.

In addition to his mathematics, his other claim to fame was the enormous effort he made to secure funding to preserve Durham Castle. The castle was in danger of sliding down the hill upon which it was built. Heawood, as secretary of the Durham Castle Preservation Committee, battled to raise the funds necessary to save this historic building. His work was rewarded by the University in 1931 who conferred a Doctorate of Civil Laws, by Durham County Council, and by the government in 1939 by the award of Officer of the Order of the British Empire (OBE).

Heawood was described as an eccentric, with a large moustache that earned him the nickname *Pussy*. His obituary by the London Mathematical Society described him:

In his appearance, manners, and habits of thought Heawood was an extravagantly unusual man ... He usually wore an Inverness cape of strange pattern and manifest antiquity, and carried an ancient handbag. His walk was delicate and hasty, and he was often accompanied by a dog, which was admitted to his lectures.

Eccentric he may have been, but he was held in affection, particularly in Durham where he lived for nearly seventy years and where he died at the age of 93.

Heffter, Lothar Wilhelm Julius ³

Born 11 June 1862: Köslin, Germany

Died 1 January 1962: Freiburg im Breisgau, Germany

Lothar Heffter studied mathematics and physics at both Heidelberg and Berlin Universities. After graduating, he taught at Giessen, Bonn, Aachen and Kiel Universities, before taking up a position at Freiburg University in 1911. He remained in Freiburg until his death just six months short of his centenary. Although appointed emeritus professor in the mid-1930s, he continued to lecture part time, returning to full time teaching during World War II, due to the call-up of other teaching staff. His teaching career came to an end with the destruction of Freiburg by allied bombing in 1944.

He published a paper in 1891 dealing with the colouring of maps on closed orientable surfaces, and two further graph theory papers, *Über nachbarconfigurationen, tripelsysteme und metacyklische gruppen* in 1896 and *Über metacyklische gruppen und nachbarconfigurationen* in 1898. In old age, he published two autobiographies — *Mein Lebensweg und meine Mathematische Arbeit* in 1937, and *Beglückte Ruckschau auf 9 Jahrzehnte* in 1952. He also published work on analytical geometry and linear differential equations.

³ It has not been possible to obtain a photograph of Heffter.

Kempe, Alfred Bray

Born 6 July 1849: Kensington, London, England

Died 21 April 1922: London, England

Alfred Kempe was born the third son of the Rector of St James' Church, Piccadilly. He won a Camden Exhibition award that enabled him to enter Trinity College, Cambridge, to study mathematics where he was taught by Cayley. He graduated in 1872



with a BA and although his main two loves were mathematics and music he chose, on leaving Cambridge, to become a barrister, with mathematics and music remaining lifelong hobbies. He was called to the bar in 1873, and in 1909 became a Bencher of the Inner Circle on the western circuit. He became an authority on ecclesiastic law and during his career held many appointments as Chancellor, a lay office of legal advisor to an Anglican diocese. These included Newcastle, Southwell, St Albans, Peterborough, Chichester and Chelmsford, and in 1912 the most important chancellorship of all, that of the diocese of London.

In 1879 Cayley and Sylvester (and others) proposed Kempe for election to the Royal Society, and his fellowship, mainly for his work on mathematical linkages, was conferred in June 1881. He was elected to the Council of the Royal Society in 1897 and the following year was appointed the Society's Treasurer; his service was recognised by a knighthood in 1912.

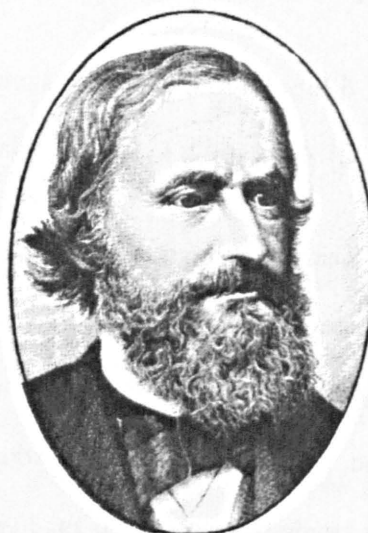
Also in 1879 Kempe published his celebrated (but fallacious) proof of the four-colour conjecture. However to remember him solely because of this error is to do him an injustice as his paper provided the foundations upon which future mathematicians based their work.

He was described as having great personal charm, was modest, urbane, frank with common sense mixed with a little humour. He was thorough in everything he did and was an accomplished peacemaker. He was an accomplished mountaineer and amongst his many honours was the naming of Mount Kempe and Kempe Glacier in Antarctica, after him, reward for his contributions to expeditions to that continent.

Kirchhoff, Gustav Robert

Born 12 March 1824: Königsberg, Germany

Died 17 October 1887: Berlin, Germany



Gustav Robert Kirchhoff was born in Königsberg, where he was educated at the Albertus University.

While he was there he formulated and published his famous laws of electrical circuits. He graduated in 1847 and moved to Berlin. After graduating, he spent three years at the University of Berlin where

he earned his doctorate with a dissertation based on his laws of electrical circuits. In 1850, he became professor of physics at Breslau at the age of 26; in the same year he solved a problem concerning the deformation of elastic plates.

The mathematics that Kirchhoff employed in his laws of electrical currents was to become applicable to what was to become graph theory, as it was the first algebraic approach to the subject. In 1854 he settled in Heidelberg, where he developed his fundamental law of electromagnetic radiation. He worked on the nature of electrical currents, proving that the velocity of a current is independent of the character of the wire and approximated the speed of light. He also studied black body radiation and the spectrum of the sun, and discovered two chemical elements, caesium and rubidium.

In 1876, he became professor of theoretical physics at the University of Berlin. This suited Kirchhoff as for many years, due to a disability, he was forced to use crutches or a wheelchair, and his increasingly failing health inhibited his experimental work. The move allowed him to concentrate on teaching and theoretical research. Amongst other honours, he was elected a fellow of the Royal Society of Edinburgh in 1868 and of the Royal Society in 1875. In addition, he was awarded the Royal Society's Rumford Medal in 1862 for his work on spectroscopy, including his three laws of spectroscopy, and the lunar crater Kirchhoff was named after him.

Kirkman, Thomas Penyngton

Born 31 March 1806: Bolton, Lancashire,

Died 3 February 1895: Bowdon, Lancashire,



Thomas Kirkman attended a grammar school in Bolton, being taught Greek and Latin, and showed such potential that his masters believed he would benefit from a university education. His father decided that he should leave school at the age of 14 and join the family business where he worked for nine years. In his leisure time he continued to study languages, adding French and German. Contrary to his father's wishes, he entered Trinity College, Dublin, to study mathematics, philosophy, the classics, and science. Returning to England with his BA he entered the Church of England in 1835. He had two brief curacies before being appointed as the rector of the parish of Croft with Southworth in Lancashire in 1845, where he remained until 1892.

His first mathematics paper was published in 1847 in which he showed the existence of so-called *Steiner systems*, some 7 years before Steiner posed the question of whether such systems exist; this is another instance of the originator not receiving due recognition of discovery. However, his name has become associated with the fifteen schoolgirls problem which he posed in 1850:

Fifteen young ladies of a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk abreast more than once.

The solution to this type of problem involves constructing a so-called *resolvable design*.

During his lifetime, he published a considerable number of both theological pamphlets and mathematical papers. His mathematical output included work on combinatorial puzzles, generalisations of quaternions, geometry and points of congruence of Pascal lines, this latter work being included in standard texts. In graph theory he studied enumeration of polyhedra and cycles on polyhedra. At the age of 78, he published his first paper on knots, and went on to collaborate with Tait, producing a number of papers and tables on the classification of knots with eight, nine and ten crossings. He was elected a Fellow of the Royal Society in 1857.

Lewis, Jr. Daniel Clark ⁴

Born 14 August 1904: Millview, New Jersey, USA

Died 19 June 1997: Baltimore, Maryland, USA

Lewis was awarded AB and AM degrees from Haverford College, Pennsylvania, in 1926 and 1928, and in 1932 he received a PhD from Harvard for his dissertation titled *Infinite Systems of Ordinary Differential Equations with Applications to Certain Second Order Non-Linear Partial Differential Equations of Hyperbolic Type* under the supervision of G. D. Birkhoff. He was a National Council Research Fellow from 1933 to 1935, before going on to become an instructor in mathematics at Cornell University, and assistant professor, then associate professor at the University of New Hampshire, before moving to the University of Maryland in Baltimore, in 1946. He worked from 1943 to 1945 at the War Research Establishment at Columbia University. In 1948, he became Professor of Applied Mathematics at The Johns Hopkins University, becoming Emeritus Professor in 1971, and from 1958 to 1959 he carried out research at Princeton's Institute for Advanced Study.

He was a consultant to industry and government, contributed many articles to (and undertook reviews for) mathematical journals, and was editor of the *American Journal of Mathematics* from 1949 to 1952. His research topics included ordinary differential equations in the real domain (the set of real numbers), dynamical systems, and partial differential equations, as well as the four-colour map problem. He discovered and developed the theory of 'autosynartetic' solutions, generalising a theory of Poincaré on periodic solutions of ordinary differential equations.

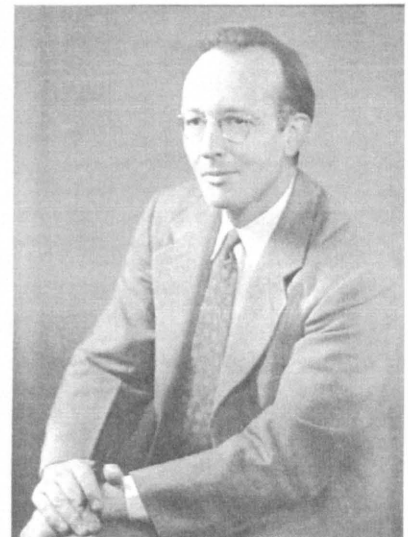
Around the time of World War II Lewis collaborated with G. D. Birkhoff, and in 1946 their lengthy and impressive paper *Chromatic polynomials* was published. This paper was an attempt to bring together all previous work on map colouring and to offer some of their own conjectures.

⁴ It has not been possible to obtain a photograph of Lewis

MacLane, Saunders

Born 4 August 1909: Taftville, Connecticut, USA

Died 14 April 2005: San Francisco, California, USA



Although christened Leslie Saunders MacLane, his parents came to dislike the name Leslie so it was dropped. He enrolled at Yale University in 1926, obtaining a bachelors degree in physics in 1930, an

MA whilst a fellow at the University of Chicago in 1931, and a PhD in mathematics from the University of Göttingen in 1934 for his thesis *Abbreviated Proofs in the Logical Calculus*. At Chicago MacLane was strongly influenced by Eliakim Hastings Moore, and accepted Moore's invitation to undertake graduate research at Göttingen. He was the Peirce instructor of mathematics at Harvard University from 1934 to 1936, then instructor at Cornell and Chicago Universities, each for a year, before joining the mathematics faculty at Harvard as assistant professor in 1938, becoming a full professor in 1947.

In 1941, he collaborated with Garrett Birkhoff producing a significant book, *A Survey of Modern Algebra*. This was instrumental in modernising the teaching of algebra in colleges and universities. MacLane also worked on logic, valuations, and their extensions to polynomial rings, and category theory. He carried out valuable war work as part of the Applied Mathematics Group at Columbia University during World War II, and corresponded with Whitney. Between 1935 and 1937, he published three papers on graph theory which included extensions to Whitney's work.

He was the recipient of many honorary degrees, including doctorates from Purdue, Yale and Glasgow Universities. In 1989 he was awarded America's highest award for scientific achievement, the National Medal of Science, as well as being awarded the Chauvenet Prize and the Distinguished Service Award of the Mathematical Association of America, and the Steele Career Prize of the American Mathematical Society. He was President of the American Mathematical Society from 1973 to 1974.

Moore, Eliakim Hastings

Born 26 January 1862: Marietta, Ohio, USA

Died 30 December 1932: Chicago, USA

Moore attended Woodward High School from 1876 to 1879, and his love of mathematics and astronomy was triggered whilst working for the director of the Cincinnati Observatory during one summer's vacation. He enrolled at Yale at the age of 17, earning his A.B degree in 1883 and his doctorate two years later.



He was encouraged by his supervisor to travel to Germany to continue his studies, where during the academic year 1885–86 he attended the Universities of Göttingen and Berlin. On returning to the USA in 1886, Moore became a high school instructor for a year, and then a tutor at Yale for two years, before taking up a permanent position at Northwestern University in 1889. In 1892, he was appointed professor of mathematics and acting head of the Department of Mathematics at the new University of Chicago, a great honour for such a young scholar, becoming head in 1896. He served as Vice-President of the American Mathematical Society from 1898 to 1900, was elected President in 1901, and Colloquium Lecturer in 1906. He received honorary degrees from Göttingen, Yale, Clark, Toronto, Kansas, and Northwestern Universities.

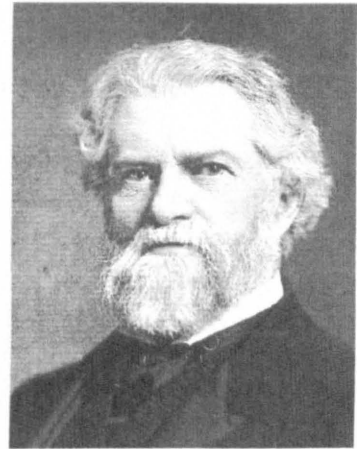
Moore's main areas of work were in algebra, groups and the foundations of geometry. In later years, he worked on the foundations of analysis. Although he did not publish any graph-theoretical work, he was the doctoral supervisor to both Veblen and Birkhoff. In later years, he worked on the foundations of analysis. Raymond Clare Archibald summed up Moore as follows [4]:

Moore was an extraordinary genius, vivid, imaginative, sympathetic, foremost leader in freeing American mathematics from dependence on foreign universities, and in building up a vigorous American School, drawing unto itself workers from all parts of the world.

Newcomb, Simon

Born 12 March 1835: Wallace, Nova Scotia,
Canada

Died 11 July 1909: Washington DC, USA



Simon Newcomb was Hassler Whitney's maternal grandfather, and his most distinguished forebear.

With little formal education, except that taught by

his itinerant schoolteacher father, he was apprenticed to an herbalist at age 16. His employer turned out to be a charlatan, so Newcomb walked out on his job and, with no money in his pocket, walked 120 miles from Salisbury to the port of Calais, in Maine, and worked his passage on-board ship to Salem, Massachusetts, where he joined his father who had earlier moved to the United States.

Newcomb taught in high schools and worked as a private tutor, whilst continuing his studies in his spare time. He joined the American Nautical Almanac Office, studying part-time at the Lawrence Scientific School, supervised by Benjamin Pierce, and graduated with a BS in 1858. In 1861, he became professor of mathematics and astronomer at the Naval Observatory in Washington. For his astronomical work on the positions of Uranus and Neptune, the Royal Astronomical Society awarded him its Gold Medal in 1874, presented by Arthur Cayley. In 1877, Newcomb became director of the American Nautical Almanac Office and, along with his increasingly senior position within academia; he was promoted to the rank of Rear Admiral. In 1884, he was appointed professor of mathematics and astronomy at Johns Hopkins University, a post which he held until 1893. Much honoured, he served as President of the American Mathematical Society from 1897 to 1898. The following geographical features are named after him; Cape Newcomb on the Hoyt Islands, Greenland; the Lunar crater Newcomb; the Martian crater Newcomb; and the minor planet #855 Newcombia. Additionally, a US Navy surveying ship was named the USS Simon Newcomb. In the autumn of 1908 he became ill, diagnosed with cancer of the bladder. He was buried with military honours in Arlington National Cemetery, a ceremony attended by President William Taft and representatives from several foreign governments.

Peirce Benjamin

Born 4 April 1809: Salem, Massachusetts, USA

Died 6 October 1880: Cambridge,
Massachusetts, USA



Benjamin Peirce's father was a state legislator in Massachusetts, as well as being a librarian at Harvard. The young Benjamin was educated at Salem Private Grammar School and entered Harvard in 1825, aged 16. He graduated in 1829,

becoming a teacher for two years before being appointed a tutor at Harvard, where in 1833 he became professor of mathematics and natural philosophy. Also in 1833 he received a Master's Degree at Harvard, and although he did not earn a doctorate, the frontispiece of the first issue of the *Journal of Mathematics* records that he was an LL.D (probably an honorary degree). In 1842, he became Perkins Professorship of Mathematics and Astronomy, a position he held until his death.

During the early part of his career, Peirce wrote and published a number of textbooks. Although these books were well written and contained elegant mathematics, they were considered too demanding for students of that time. He also received some criticism of his style of lecturing, which many students found difficult to follow: only his more able pupils were equipped to appreciate his enthusiasm for mathematics and to benefit from it. He contributed to the determination of the orbit of Neptune, and calculated the perturbations of Neptune on the orbits of Uranus and other planets; this led to his appointment as Director of Longitude Determination for the US Coastal Survey in 1852, and then Director of the Survey from 1867 to 1874.

Peirce explored a wide range of research topics, and was instrumental in providing the educational structure that would encourage mathematicians of America in research and have a considerable influence on many of those who would develop the subject in the USA. During his time at Harvard, Peirce was influential in elevating the status of the college to that of a leading national institution. He was the leading mathematician and astronomer in the US and is regarded as having made the first important American mathematical research contribution.

Peirce, Charles Sanders

Born 10 September 1839: Cambridge,
Massachusetts, USA

Died 19 April 1914: Milford, Pennsylvania, USA



As a young boy, C S Peirce thrived on the intellectual atmosphere prevailing at the family home, where his father, Benjamin Peirce, entertained academics, politicians, poets, scientists and mathematicians. Although this provided a scholastic environment there was a disadvantage to

the way he was raised; his father avoided discipline, fearing that it might inhibit independence of thought. This indulgent attitude provided a platform where he could show off his undoubted genius, but it left him not knowing how to behave and interact with people. This lack of parental guidance made it difficult for him to fit in to society, and led to a problematic future life [5].

He enrolled at Harvard College at age 15, but he did not shine in his work, his preference being to study on his own with books of his own choosing. He graduated with an AB in 1859, and then entered the Lawrence Scientific School, being under the influence of his father and meeting with greater success than in his undergraduate years. He received a Master's degree from Harvard in 1862 and a ScB from the Lawrence Scientific School 1863. He remained at Harvard doing graduate research, and in the spring of 1865 he presented the Harvard Lectures on *The Logic of Science*.

Peirce's interests and areas of research were extremely wide ranging, including probability and statistics, which he utilised in his philosophical views and scientific methods. His research included mathematical work, psychophysics (or experimental psychology) and species classification. He also spent considerable time on the four-colour problem, on one occasion claiming that he had found a solution [6].

From 1859, in parallel with his academic career, Peirce held a position as a part-time assistant at the Coast Survey for nearly thirty years; some of his time with the Survey was under his father as director. In 1876, Peirce produced one of his most notable inventions, the *Quincuncial Map*

Projection, published in the second volume of the *American Journal of Mathematics* in 1879; this earned him the reputation as one of the greatest map-makers up to that time. Although his invention was not taken up at that time, it was being used in the middle of the twentieth century to display air routes.

In 1879, he obtained the position of part-time lecturer in logic at Johns Hopkins University under Sylvester, but due to marital difficulties this appointment lasted only a few years. Thereafter his only steady work, and therefore income, was from his part-time work with the Coast Survey. He became increasingly quarrelsome and distanced from his superiors, working in isolation at a time when the Survey was experiencing a lack of funding, and in 1890 his long-awaited major report was submitted to the Survey — but they declined to publish it without considerable revision. This Peirce failed to do, and at the end of 1891 the Survey ran out of patience and requested his resignation. This left him with no regular income. Much of his work after 1890 was either rejected for publication or incomplete.

Although he was not accorded the undoubted recognition he deserved during his lifetime, there is now a growing interest in his work, especially in logic. Some believe that he was the greatest original intellect to have been born on the American continents [7]. To gauge how his contemporaries saw Peirce, it is pertinent to quote Thomas Scott Fiske who, in recalling the early days of the American Mathematical Society, described Peirce in the early 1890s [8]:

His dramatic manner, his reckless disregard of accuracy in what he termed “unimportant details”, his clever newspaper articles describing the meetings of our young Society interested and amused us all. ... He was always hard up, living partly on what he could borrow from friends, and partly on what he got from odd jobs ... He was equally brilliant, whether under the influence of liquor or otherwise, and his company was prized by the various organisations to which he belonged; and he was never dropped from any of them even though he was unable to pay his dues.

Indeed, for much of his adult life he lived like a social outcast, sometimes even stealing to eat, and occasionally being without a permanent address. He believed that there was an international plot conspiring to undermine and destroy him. His continuing dream was to generate vast wealth from amazing inventions, but this never happened. Peirce died of cancer in his seventy-fifth year, in isolation on his farm.

Petersen, Julius Peter Christian

Born 16 June 1839: Sorø, Denmark

Died 5 August 1910: Copenhagen, Denmark

Julius Petersen, the son of a dye worker, attended the Sorø Academy School and it was here he began his interest in mathematics, particularly in problem solving. He spent considerable time trying to solve



the classic problem of trisecting an angle with rule and compasses. Although wanting to provide their son with a good education, due to lack of finance, his parents were forced to take him from the Academy. He went as an apprentice to his uncle, a grocer in Kolding, Jutland, who died a year later and left the young Petersen with sufficient money for him to return to Sorø and continue his studies.

He entered the College of Technology in Copenhagen in 1856, where he published his first paper, on logarithms. In 1860, he passed his civil engineering examinations, but having used up his inheritance he was unable to support himself through university. So he took a position at a private school from 1859 to 1871, together with other teaching jobs to increase his income. He entered the University of Copenhagen in 1862, graduating with a Master of Mathematics in 1868, and was awarded his doctorate in 1871. He returned to the College of Technology as a dozent, where he taught until 1887, with spells of teaching at a polytechnic and a military academy. In 1877, he became professor of mathematics at the University of Copenhagen, a position that he held until his retirement.

Petersen's most notable work was in geometry, and his graph-theoretical papers would be taken up by a number of mathematicians in the USA. His name is remembered for the *Petersen graph* and he also published an important paper on graph factorisation. He published a number of highly regarded school and college texts. Some of his research topics were from algebra and geometry, number theory, analysis, differential equations, and mechanics. Additionally he published work on mathematical physics, mathematical economics, and cryptography.

He was a founder member of the Danish Mathematical Society in 1873.

Story, William Edward

Born 29 April 1850: Boston, Massachusetts, USA

Died 10 April 1930: Worcester, Massachusetts, USA

William Story's ancestor, the Englishman Elisha Story, emigrated to America around 1700 and settled in Boston, Massachusetts. Other forebears included Dr Elisha Story, one of the citizens of Bunker Hill who took part in the Boston Tea Party.



Story entered Harvard University in 1867, graduating with honours in 1871, as one of the first students to be awarded the newly created honours degree in mathematics. He travelled to Germany, studying at both Berlin and Leipzig Universities, obtaining a PhD from the latter in 1875 for his thesis, *On the Algebraic Relations Existing between the Polars of the Binary Quantic*. On returning home he became a tutor at Harvard, and must have impressed Benjamin Peirce such that when Sylvester approached Peirce for suggestions of suitable mathematicians worthy of consideration to join the newly founded Johns Hopkins University, Peirce recommended Story. Whilst there Story became embroiled in the controversy surrounding Kempe's infamous paper on the four-color problem, and in presenting it and a follow-up paper of his own he incurred Sylvester's wrath. Story was to continue to study the four-colour problem for most of his academic life.

In 1887, Story became head of mathematics at the newly founded Clark University in Worcester, Massachusetts, where he developed a first-class PhD programme. This was a credit to him, with twenty-five doctorates awarded between 1892 and 1921, nineteen under his direct supervision. In spite of all his good work, bad luck struck Story in 1921, when supposed financial problems forced the university to close its graduate programme and he was required to resign. During his later years, Story became interested in the history of mathematics and compiled a considerable bibliography of mathematics and mathematicians; the American Mathematical Society now looks after this archive.

Sylvester, James Joseph

Born 3 September 1814: London, England

Died 15 March 1897: London, England

James Joseph's father, a merchant, was named Abraham Joseph. In his teens James Joseph added the surname Sylvester, as having three names was a necessary requirement for emigration to the USA (a step being taken by his brother at the time).



In 1828, at the age of 14, he entered the non-sectarian University College, London, where he was taught by De Morgan. After five months, his family decided to withdraw him. However they were keen for him to benefit from a university education, so they sent him to study at the Royal Institute in Liverpool. In 1831, he went to St. John's College at Cambridge University, although he suffered from a lengthy illness that caused him to miss most of the academic years 1833–35. Although a brilliant scholar, coming second in the highly valued Mathematical Tripos in 1837, he was not permitted to receive his degree, because he was Jewish and unwilling to sign up to the Thirty-Nine Articles of the Church of England. However, he did obtain BA and MA degrees from Trinity College in Dublin in 1841. At either Cambridge or Oxford, or indeed at any sectarian institution, he was similarly unable to obtain a university position that his undoubted ability deserved. This was due to the conflict between his religious ancestry and the requirements of the statutory Tests Act, which was in force in Britain at that time and which was not rescinded until 1871. From 1837 to 1841, he was Professor of Natural Philosophy at one of the few non-sectarian institutions, University College, London.

In 1841, he became professor of mathematics at the University of Virginia, in the United States of America, his application being strongly supported by, amongst others, De Morgan. His tenure lasted only a few months, when he resigned due to a clash with a student and Sylvester's perception of the lack of support from the University in the matter.

Unable to obtain a suitable post in America, he reluctantly returned to England where he gained employment for eleven years as an actuary at the Equity and Law Life Assurance Company in London; he also gave private lessons in mathematics. In 1846, he decided to study law and during his training as a barrister he met Cayley. Sylvester was an active researcher and published many papers during his career. He gave lectures at the Royal Institution, and on one occasion a member of his audience was Kempe, who worked with him on mechanical linkages, making significant discoveries. In 1855, Sylvester became professor of mathematics at the Royal Military Academy at Woolwich, where he remained until 1870. He was reluctantly obliged to leave this employ, as new War Office regulations made it compulsory for teaching staff at military academies to retire at the age of fifty-five.

He was head-hunted by The Johns Hopkins University's President, Daniel Coit Gilman. So Sylvester returned to America in 1876, becoming the first professor of mathematics at Johns Hopkins, where he enjoyed considerable success [9]. Finally, in 1883, at sixty-eight years of age, he was recognised in his native land. He applied for, and was offered, the Savilian Chair of Mathematics at Oxford University, a position he held until his death. In his late 70s, suffering from lapses of memory and partial blindness, he returned to London with a deputy appointed to cover his duties in Oxford.

Like Cayley, he studied trees and their connection to chemical composition. While at Johns Hopkins he published a paper in which the word *graph* (in the sense of graph theory) was used for the first time, in a chemical context. He was familiar with the early days of the quest for a solution of the four-colour problem, having commissioned and published Kempe's paper that contained a false proof.

Sylvester was awarded a number of honours and prizes, including election as a Fellow to the Royal Society in 1839 at the age of 25, and was the recipient of the Royal Society's Royal Medal in 1861 and Copley Gold Medal in 1880. The lunar feature *Crater Sylvester* was named in his honour.

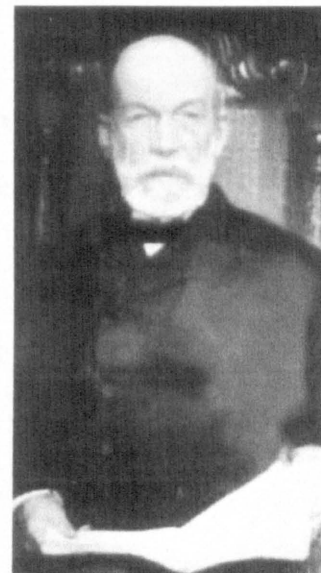
An unpredictable, erratic and flamboyant scholar, he could be brilliant, quick tempered, and restless, filled with immense enthusiasms and an insatiable appetite for knowledge. Throughout his life, he fought for the underdog in society. He supported education for the working classes, for women, and for people who were discriminated against.

Tait, Peter Guthrie

Born 28 April 1831: Dalkeith, Scotland

Died 4 July 1901: Edinburgh, Scotland

Peter Guthrie Tait was six when his father, secretary to Walter Francis Scott (fifth duke of Buccleuch) died. The family moved to Edinburgh to live with an uncle. Tait was influenced by his uncle's enthusiasm for science, and was enrolled



in the Edinburgh Academy at the age of only 10. He topped his class every year at the Academy, focusing on classics rather than science, but became interested in mathematics during his fourth year and showed much promise. In 1846 he won the mathematics Edinburgh Academical Club Prize, beating James Clerk Maxwell (1831–1879) into third place; however, the next year Maxwell was to triumph, with Tait coming second. Maxwell and Tait entered the University of Edinburgh in 1847 when Tait was 16, but after a year Tait moved to Peterhouse in Cambridge University, graduating as senior wrangler in the Mathematics Tripos and first Smith's prizeman. He remained there as a fellow, before moving to Queen's College, Belfast, as professor of mathematics in 1854. In 1859 he took the Chair of Natural Philosophy at the University of Edinburgh.

Tait studied the four-colour problem; indeed he published two papers that attempted to provide simpler explanations to parts of Kempe's false proof. He suggested an alternative approach to the problem, which has become known as *Tait colouring* or *edge colouring*.

Tait wrote some 360 papers and 22 books on many diverse topics and together with William Thomson (Lord Kelvin) (1824–1907) published a highly influential book on natural philosophy. Although a successful scientist and scholar, he was also argumentative, having a number of public heated disputes with fellow scientists. He was a strong supporter of the Royal Society of Edinburgh and a founder member of the Edinburgh Mathematical Society in 1883. Although never elected to the Royal Society, he was awarded their Royal Medal in 1886. Tait was eccentric, dressed poorly, and avoided dining out or putting on a dress suit. He died shortly after he retired.

Tietze, Heinrich Franz Friedrich

Born 13 August 1880: Schleinz, Austria

Died 17 February 1964: Munich, Germany

Heinrich Tietze was son of Emil Tietze, the Director of the Geological Institute at the University of Vienna. He studied at the Technische Hochschule in Vienna from 1898 to 1902, and then in Munich for a year, returning to Vienna where he received his doctorate in 1904.



He became Privatdozent in Vienna in 1908, and between 1910 and 1919 was a professor at the University of Brunn (now Brno). He held professorships at the University of Erlangen from 1919 to 1925, and at the University of Munich from 1925, becoming emeritus professor in 1950. World War I interrupted his academic career, when he served in the Austrian Army from 1914 to 1918. His time at Munich coincided with the Nazi years in Germany, which made life difficult for him, although his position was reasonably safe.

Tietze contributed to the beginnings of algebraic topology and formulated theories on the subdivisions of cell complexes. Most of his books and some two hundred papers were written during his time at Munich. He is mostly remembered for his *Tietze transformations*, which

... change one presentation of a finitely presented group to another presentation without changing the group which is defined by the presentation.

Within topology, he worked on knot theory, Jordan curves, and continuous mappings of areas as well as combinatorial group theory. In addition, his research embraced rule and compasses constructions, continued fractions, partitions, the distribution of prime numbers, and differential geometry. In graph theory he wrote on the colouring of maps on non-orientable surfaces. He also wrote a well-known book, *Famous Problems of Mathematics*. Tietze received numerous honours for his contributions to mathematics, including his elections as a Member of the Bavarian Academy of Sciences and Fellow of the Austrian Academy of Sciences.

Tutte, William Thomas

Born 14 May 1917 in Newmarket, Suffolk,
England

Died 2 May 2002 in West Montrose, Ontario,
Canada

Bill Tutte showed early academic promise and developed an interest in astronomy. At age 11 he entered the Cambridge and County High School for Boys. He won many academic prizes, which he treasured: they were still to be found in his office at the end of his life. In 1935, he entered Trinity



College, Cambridge, to study Natural Sciences, specialising in Chemistry and graduating with a first-class honours degree. He joined the Trinity Mathematical Society and became firm life-long friends with three mathematics students — Leonard Brooks, Cedric Smith and Arthur Stone. They published a paper in 1940 (although by that time Tutte was undertaking war work) on *squaring the square* — the dividing of a square into unequal smaller squares; although a recreational problem, it had links with electrical networks, and led to other mathematical applications.

Shortly after starting his postgraduate study in chemistry, and after the outbreak of the Second World War, he was approached by his tutor and asked to offer his services to the highly secret organisation at Bletchley Park, the wartime headquarters of the British Government Code and Cipher School. Here Germany and her allies' airborne communications were monitored. Cryptographers had met with considerable success with reading the German naval and air force codes, but the one used by the army, FISH, proved to be more difficult to analyse. Tutte was asked to try to unravel the army machine-cipher, and with brilliant ingenuity over a period of four months deduced the entire machine without ever having seen one. This was a major contribution to the success of the allies in World War II, an achievement of which Tutte could be rightly proud, being part of the work that shortened the war and saved allied lives. After the war Tutte returned to academic life in Cambridge. He was elected to a Research Fellowship at Trinity College gaining a doctorate in 1948 for a thesis on graph theory, *An Algebraic Theory of Graphs*.

Following an invitation from H. S. M. Coxeter he emigrated to Canada, joining the University of Toronto in 1949 as a lecturer, and later as an associate professor. In 1962, he became Professor of Mathematics at the new University of Waterloo where he remained until his retirement, becoming Professor Emeritus. At Waterloo Tutte founded the world-famous Department of Combinatorics and Optimization and was involved in the foundation of the *Journal of Combinatorial Theory*. In 1958, he was elected a Fellow of the Royal Society of Canada, awarded the Society's Tory Medal in 1975, and elected Fellow of the Royal Society in 1987. In October 2001, he was inducted as an Officer of the Order of Canada.

He published many papers and books including a seminal paper in 1959 in which he developed the work of Whitney on matroids and graphs and also the jointly published paper in 1973, with Whitney, which was an attempt to give a broad explanation for an approach to the four-colour problem; their intention to clarify the situation that existed in 1972.

Having signed the Official Secrets Act in 1941 he felt obliged to remain silent regarding his work at Bletchley Park. However, in the mid-1990s the significance of Bletchley Park entered the public domain relieving him of the burden of remaining silent — but even then, he was not keen to discuss openly that period of his life. To sum him up, nothing could be better than a quotation from one of his neighbours, Jeremy Humphries, from West Montrose, a fitting tribute to a good and great man [10]:

Bill was an old neighbour of mine; he was kind enough to play chess with me when I was a teenager (and he was in his late 50's). Years later, when I discovered in Singh's book that he was one of the masterminds from Bletchley Park (something that none of his neighbours, nor his University of Waterloo colleagues knew in the early 1970's), I felt less badly about being regularly whupped by my kind neighbour. On discovering a whole new side to Bill (back in the 1970's, I thought of him as one of the world's leading experts on the Four Colour Map Problem), I tracked him down ... Even today, I continue to be amazed that he built an entire "new" career around graph theory, never able to claim credit for his remarkable intellectual accomplishment as a 24-year-old in World War II. Back in the early 1970's many believed that Bill would be the first to prove the Four Colour Map Conjecture.... Bill was "robbed" of a (small) piece of his graph theory glory by a descendant of the Colossus machine that was built to implement his pencil-and-paper attack on the Lorenz cipher.

Veblen, Oswald

Born 24 June 1880: Decorah, Iowa, USA

Died 10 August 1960: Brooklin, Maine, USA

Oswald Veblen's paternal grandparents emigrated from Norway to Wisconsin in 1847. One of their children, Andrew Anderson Veblen (1848–1932), Oswald's father, became professor of mathematics and physics at the University of Iowa. Oswald entered that university in 1894, graduating with an



AB in 1898. He then went to Harvard for one year, earning a second AB in 1900, and then spent three years at the University of Chicago, gaining a PhD in 1903 for his thesis, *A System of Axioms for Geometry*. In 1905 he joined the mathematical faculty at Princeton University.

During World War I Veblen became a Major in the Ordnance in charge of firing and ballistic work at the Aberdeen Proving Grounds, Maryland — a position that he retained throughout World War II. Back at Princeton, after the war, Veblen quickly became regarded as a leading geometer, and because of his work many graduate students applied to study or to be employed there. He assisted the University's Dean, Henry Burchard Fine, to develop the department and recruit other distinguished mathematicians. In addition, his research and influence ranged over many areas of mathematics, including the foundations of geometry and topology, relativity theory and symbolic logic. Through the work of Veblen and his students, Princeton became one of the leading centres of topology; as a result, he earned the rare designation 'statesman of mathematics' around the world, a description found in many articles on Veblen [11]. Later he became interested in differential geometry, and from 1922 most of his publications were on this subject and its connections with relativity. Although few in number, his papers on graph theory, and particularly his 1916 Colloquium lectures, were very influential and were considered for many years 'the best introduction to the subject' [12].

Veblen was awarded many honorary degrees by American and overseas universities, and as he was a descendant of Norwegian grandparents, Norway conferred on him the award of Knight of the Royal Order of St Olaf, an honour that had also been conferred on his father.

After the rise of Adolf Hitler in Germany, Veblen was much involved in the recruitment into the American academic world of many notable foreign mathematicians. This earned him considerable respect and not a little gratitude. He was an active member of the American Mathematical Society, being Vice-President in 1915, Colloquium Lecturer for the society in 1916, and President in 1923–24. The Henry Burchard Fine Professorship, the first mathematics research chair in the USA, was endowed at Princeton in 1926 and Veblen became its first incumbent. In the academic year 1928–1929 he taught at Oxford University as part of an exchange with G H Hardy. In 1932, he resigned this position at Princeton University to become the first professor and chairman of the faculty of mathematics at the recently established Institute for Advanced Study at Princeton, becoming Emeritus Professor in 1950.

During his last years he was partially blind, although he retained some peripheral vision. He worked to develop aids, for himself and others, to assist reading and one of his ideas was taken up and produced by the American Foundation for the Blind. Towards the end of his life, he was diagnosed with a heart problem, and although he continued to pursue his interests, he died in 1960 at his summer home.

There are many testimonies to Veblen, including the authoritative tribute to him by the faculty and trustees of the Institute for Advanced Study, who wrote [12]:

We are acutely conscious of the loss to the Institute and to the world of learning of a major figure.

Oswald Veblen was of great influence in developing the Institute as a centre for postdoctoral research ... He loved simplicity and disliked sham. He placed the standing of the Institute ahead of his personal convenience. He possessed the art of friendship, and his assistance was decisive for the careers of dozens of men.

Veblen, one of the most influential mathematicians of the early twentieth century, will be remembered for his undoubted scholarly contributions to the subject and for his work in the development of mathematics, Princeton, and American scholarship in general.

Whitney, Hassler

Born 23 March 1907: New York City, USA

Died 10 May 1989: Princeton, New Jersey,
USA



Included in a potted biography of William Dwight Whitney, philologist and one of the foremost Sanskrit scholars of the second half of the nineteenth century, and Hassler's paternal grandfather, was the following sentence:

Through both parents he was descended from New England stock remarkable alike for physical and mental vigour; and he inherited all the social and intellectual advantages that were afforded by a community noted, in the history of New England, for the large number of distinguished men it produced.

From such communities, like the educated and relatively comfortably off New Englanders, came mentally and physically strong men who made considerable contributions, not only to their own geographical area, but also, in many cases, to the wider world. This can also be said to be true of the men from old England and, as will be seen, true of the ancestry of Hassler Whitney.

Whitney's paternal ancestry can be traced back to Sir Baldwin Whitney of Whitney in Herefordshire, England who was born in the first half of the fourteenth century. From the sixteenth century, the hereditary knighthood was inherited by a branch of the family that did not lead to Hassler Whitney and in 1635 ancestors of his, John and Elinor Whitney emigrated to America. Once there the family thrived and produced some notable citizens, including a number of scholars, over the next three hundred years or so.

Both his grandfathers were learned men; his maternal grandfather was Simon Newcomb, the highly acclaimed mathematician and astronomer, Rear Admiral in the United States Navy, and one of America's foremost international figures during the second half of the nineteenth century. Whitney's father was a judge.

By his own admission, Hassler's schooling was rather informal, which he appreciated. He claimed to have had little mathematics at school and none in college. His education included a period from 1921 to 1923 at a school in Switzerland, where he learned French and German. He enrolled at Yale University for undergraduate studies in 1925, and earned a PhB in physics in 1928 and a music degree in 1929. With Birkhoff as his supervisor at Harvard, he was awarded his PhD for his thesis in graph theory, *The Coloring of Graphs*, which was based on Birkhoff's 1912 paper.

Whitney produced seminal work in graph theory which was published in the years 1931–1937. Following this his interest moved to algebraic topology, where he made fundamental contributions. He made significant contributions to numerous mathematical topics — manifolds, cohomology, characteristic classes, classifying spaces, stratifications, and fibre bundles

Like many other mathematicians, Whitney contributed to the war effort in World War II. He was part of the Applied Mathematics Group and was primarily responsible for studying the use of rockets in air warfare. He also carried out liaison work with numerous Army and Navy units, and the British Air Commission. His colleague, Saunders MacLane, recalled that this liaison work with its numerous visits had 'very effective results'. In an article by MacLane he wrote of Whitney's immediate contribution and reward [13]:

George Piranian recalls it for me as follows: ... in the fall of 1943, the entire scientific staff of AMG-C gathered to witness your induction and introduction of Hassler Whitney. You described the difficulties with the mark 18 gunsight, and Hassler's quick perception and active engagement were spectacular. ... immediately after the assembly's dispersal, Hassler withdrew to his office and began writing a scientific report. A few days later, there was a question whether Hassler should be permitted to see his own report. The paper was classified, and Hassler's security clearance was held up". The clearance was eventually cleared. When AMG-C in November 1945 received a Naval Ordnance Development Award, Whitney received the first individual citation.

During a long and successful career, Whitney amassed a large number of honours and awards, including membership of many learned societies and committees and honorary doctorates. He presented the American Mathematical Society's Colloquium lectures in 1945, was Vice-President of the Society in 1948–1949, and received a number of awards; including the National Medal of Science in 1976 conferred by President Carter.

In connection with his work on the teaching of mathematics to school pupils, the mathematician and educator Anneli Lax (1922–1999) wrote [14]:

Hassler's second career occupied him during the last two decades of his life. Many of his mathematical, first-career colleagues and admirers wondered why such an incisive, original researcher would abandon his seminal work to tackle the complex intractable problems of education. And many of his second-career collaborators wondered why he gave up his prestigious academic position for activities that seemed lacking in scientific stimulation and challenge, full of potentially frustrating bureaucratic and political impediments to meaningful changes. ... These questions are still being asked, even after his death.

On another occasion, he described his attitude to research as 'a search for inner reasons' of a natural problem to which he applied his intuition to all possible alternative views, made daring guesses, and above all repeatedly tried new avenues.

Whitney's interests included music and mountaineering. He played the violin, viola and piano and was, for many years, concertmaster of the Princeton Community Orchestra. His love and prowess as an alpinist and mountaineer are legendary and surely his expertise and enthusiasm in mountaineering must have come from his maternal grandfather, Simon Newcomb, who was also a noted mountaineer and walker. The Whitney–Gilman Ridge, on Cannon Mountain in New Hampshire is named after him and his cousin. Whitney made many visits to the Swiss Alps and he is buried at Mont Dents Blanches in Switzerland.

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Appendix III

Glossary

analysis situs: the study of position or situation.

binary quantic: a homogeneous expression in two variables, such as $ax^3 + 3bx^2y + 3cxy^2 + dy^3$.

Birkhoff number: at any time, the Birkhoff number is b at time t if at time t it has been shown that a map that cannot be properly coloured with four colours must contain at least b countries.

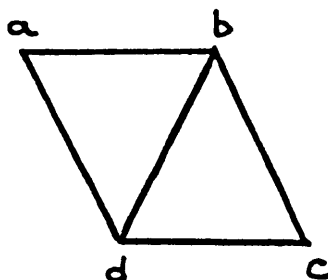
Cayley's theorem: given n labelled vertices, how many ways t_n are there of joining the vertices to form a tree — the number of labelled trees on n vertices is $t_n = n^{n-2}$, e.g. $t_4 = 16$.

chromatic number: the smallest integer k for which a map or graph can be k -coloured.

chromatic polynomial for a graph: a formula for the number of ways of colouring a graph with a given number of colours so that adjacent vertices are differently coloured.

chromatic polynomial for a map: a formula for the number of ways of colouring a map with a given number of colours so that adjacent regions are differently coloured.

closed path: a sequence of edges of a graph or map that return to the starting point — $badbcd$ is a closed path.



combinatorics: a branch of mathematics which deals with the manipulation of mathematical elements within sets that usually have a finite number of elements.

component: the separate parts of a graph; e.g. the graph illustrated has two components

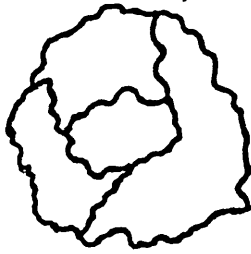


connected graph: a graph that is in one piece i.e. there is at least one path between each pair of vertices — both of the above graphs are connected graphs.

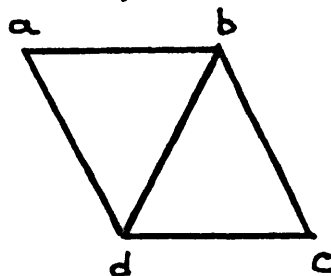
counting formula: if C_k is the number of k -sided countries in a cubic map then

$$4C_2 + 3C_3 + 2C_4 + C_5 - C_7 - 2C_8 - 3C_9 - 4C_{10} - \dots = 12.$$

cubic map: a map where exactly three edges meet at each vertex. The following is an example of a cubic map.



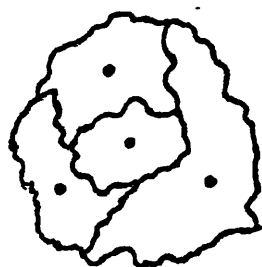
cycle: a sequence of lines on a diagram that pass only once through any point and return to the starting point [1] — $abcd$ is a closed cycle.



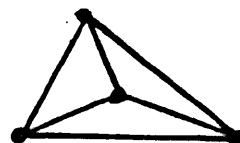
derangement: a bijection from a set into itself that has no fixed points — this is a permutation in which none of the objects appear in their natural ordered place e.g. the only derangements of $\{1, 2, 3\}$ are $\{2, 3, 1\}$ and $\{3, 1, 2\}$

D -reducible configuration: an arrangement of regions of a map where every colouring of the surrounding ring is a proper colouring, or may be converted into one by applying the method of Kempe chains.

dual graph: a dual graph is constructed by placing a vertex inside each region of a graph and joining these vertices with an edge for each common border of the original graph.



Graph with vertex placed
in each region

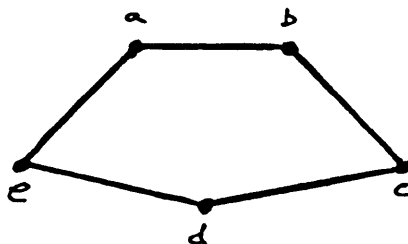


dual graph
(omitting the outside)

edge: a line joining two vertices or between two regions.

empire problem: the problem of colouring a map with several empires, each consisting of a 'mother country', and a number of 'colonies' that must be coloured the same as the mother country [1].

Eulerian path: a path that contains each edge of a graph or map once and only once and returns to its starting point; a closed path — $abcdea$ is an Eulerian path.



Euler's formula: short for Euler's polyhedron formula [1].

Euler's formula for the h -holed torus: for any map drawn on an h -holed torus,
 $(\text{number of countries}) - (\text{number of boundary lines}) + (\text{number of meeting points}) = 2 - 2h$ [1].

Euler's formula for maps on the plane or sphere: for any map drawn on the plane or sphere,
 $(\text{number of countries}) - (\text{number of boundary lines}) + (\text{number of meeting points}) = 2$ [1].

Euler's formula for the torus: for any map drawn on the torus,
 $(\text{number of countries}) - (\text{number of boundary lines}) + (\text{number of meeting points}) = 0$ [1].

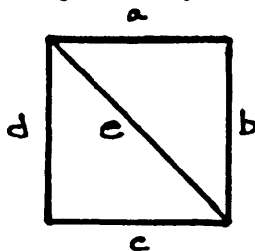
Euler's polyhedral formula: for any polyhedron,
 $(\text{number of faces}) - (\text{number of edges}) + (\text{number of vertices}) = 2$ [1].

face: a region of a planar graph bounded by edges.

four-colour problem: can the countries of every map drawn on a plane be coloured with at most four colours such that neighbouring countries are coloured differently?

four-colour problem for a sphere: can the countries of every map drawn on the surface of a sphere be coloured with at most four colours such that neighbouring countries are coloured differently?

fundamental set: a maximal set of independent cycles.

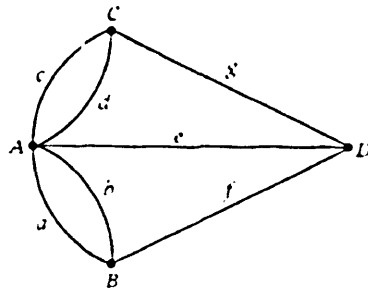


The graph shown ... contains three circuits, which we may denote by their sets of edges $C_1 = abe$, $C_2 = cde$, and $C_3 = abcd$. It is clear that the circuit C_3 is, in some sense, the 'sum' of C_1 and C_2 . To make this precise, we shall say that the **sum** of the two circuits consists of all those edges which belong to one, but not to both, of them; this can be extended in an obvious way to the sum of any infinite number of circuits. A set of circuits is said to be **independent** if no one of them can be expressed as a sum of others, so that in our example, the sets $\{C_1, C_2\}$ and $\{C_1, C_3\}$ are independent, whereas the set $\{C_1, C_2, C_3\}$ is not. A maximal set of independent circuits is called a **fundamental set**. [2]

genus of an orientable surface: orientable surfaces (2-sided surfaces) can be classified by their *genus*. An orientable surface is of genus g if it is topologically homeomorphic to a sphere with g handles. Examples of orientable surfaces are the sphere ($g = 0$) and the torus ($g = 1$).

genus of a non-orientable surface: non-orientable surfaces (1-sided surfaces) can be classified by their genus. Examples of non-orientable surfaces are the projective plane ($g = 1$) and the Klein bottle ($g = 2$).

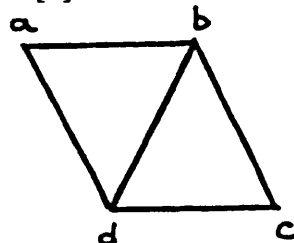
graph: a finite set of vertices, a finite set of edges, and instructions on which edges join which pairs of vertices.



This diagram is an example of what is known as a *graph*. A graph is constructed from a set of *vertices* (A, B, C , etc.), a set of *edges* (a, b, c , etc.), together with a list of which edges join which pairs of vertices.

graph theory: the study of connections between objects, a branch of combinatorics.

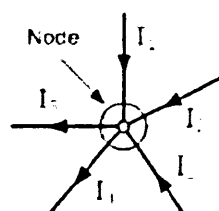
Hamiltonian cycle: a sequence of lines on a diagram that pass exactly once through every point and return to the starting point [1]. The closed cycle $abcd$ is a Hamiltonian cycle.



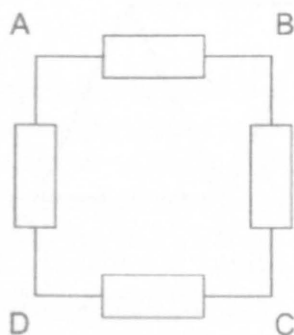
Hamiltonian graph: a graph corresponding to a Hamiltonian cycle. The above graph is a Hamiltonian graph.

Heawood conjecture: For each positive number g , there is a map on the surface of an g -holed torus that requires the integer part of $H(g) = \lceil \frac{1}{2} (7 + \sqrt{1 + 48g}) \rceil$ colours.

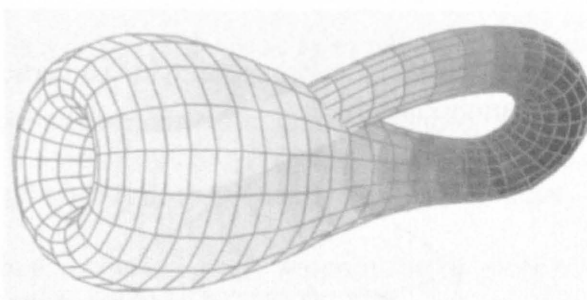
Kirchhoff's current law: the algebraic sum of the currents at any vertex of an electrical network is equal to zero. That is that the currents entering the node equal the currents leaving the node so that $I_1 + I_2 + I_3 + I_4 + I_5 = 0$



Kirchhoff's voltage law: the algebraic sum of the potential differences across all the components around any circuit (cycle) in an electrical network is zero. That is the sum of all the voltage drops around the loop equal zero so that $V_{AB} + V_{BA} + V_{CD} + V_{DA} = 0$



Klein bottle: a closed two-fold one-sided surface [2].



irreducible: If there are plane maps which need five colours, then there must be among them a map with the smallest number of regions; such a map is said to be irreducible [2].

leaf: a part of a graph that can be disconnected from the rest of the graph by removing a single edge.

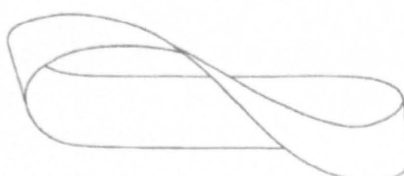
matroid: an abstract notion that generalises the idea of independence in vector spaces and graphs.

method of discharging: a procedure for determining whether a set of configurations in a map or planar graph is an unavoidable set.

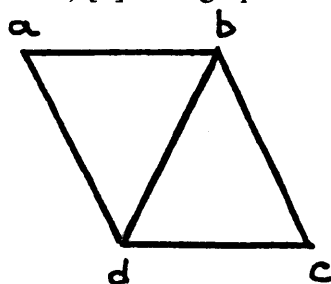
method of Kempe chains: a method of colouring maps or planar graphs in which two colours are interchanged so that regions that previously could not be coloured properly be coloured.

minimal counter-example: a map with a certain number of countries that cannot be coloured with four (or any other given number of) colours, while any map with fewer can be so coloured [1].

Möbius band: is constructed from a long rectangular strip of paper by twisting one end through 180° and then gluing the two ends together [1].



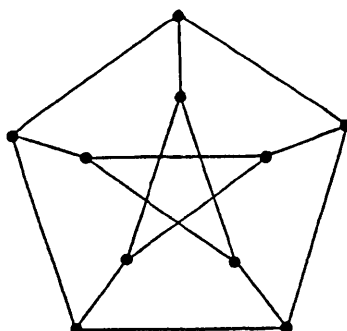
non-separable: a graph is non-separable if it is connected and cannot be disconnected by removing a single vertex (a cut vertex) [2]. The graph below is a non-separable graph.



nullity: the nullity N of a graph is defined by $N = E - R = E - V + P$, where E is the number of edges, R is the rank, V is the number of vertices and P the number of components.

partition: a partition of an integer is a representation as a sum of positive integers — for example $5 + 4 + 4 + 2$ is a partition of 15.

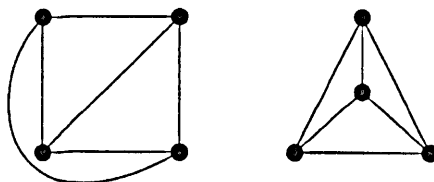
Petersen graph: The Petersen graph is a cubic graph with 10 vertices and 15 edges; it is non-planar and contains no Hamiltonian cycle.



The Petersen graph

Petersen's theorem: a regular graph of the third degree with fewer than three leaves is 3-colourable.

planar graph: a graph that can be redrawn in the plane or on the sphere in such a way that no two edges meet except at a vertex to which they are both incident. The graphs below are planar graphs.



principle of inclusion and exclusion (or sieve principle): for two subsets B and C

$$|B \cup C| = |B| + |C| - |B \cap C|.$$

The principle may be applied to more than two sets.

rank: the rank R of a graph is defined by $R = V - P$, where V is the number of vertices and P is the number of components.

reducible configuration: a configuration that cannot occur in a minimal counter-example. If a map contains a reducible configuration, then any colouring of the rest of the map with four colours can be extended (possibly after some recolouring) to a colouring of the entire map [1].

reducibility: if there are plane maps which need five colours, then there must be among them a map with the smallest number of regions; such a map is said to be **irreducible**. The basic idea is to obtain more and more restrictive conditions which an irreducible map must satisfy, in the hope that eventually we shall have enough conditions either to construct the map explicitly, or, alternatively, to prove that it cannot exist [2].

region: a general term for a country, county or state in a map [1].

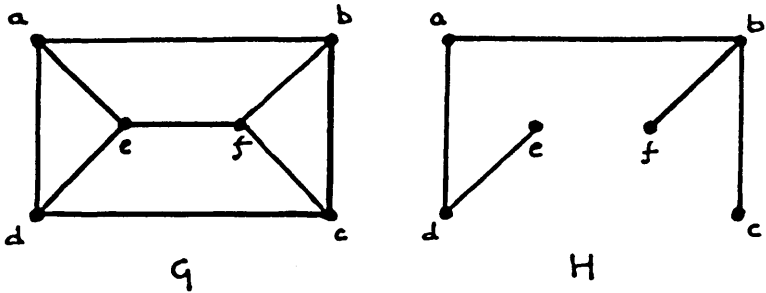
regular graph: a graph in which each vertex has the same degree — the Petersen graph is a regular graph of degree 3.

root: the root of a tree is a designated vertex from which all other vertices branch.

rooted tree: a tree in which one particular vertex is denoted as the root.

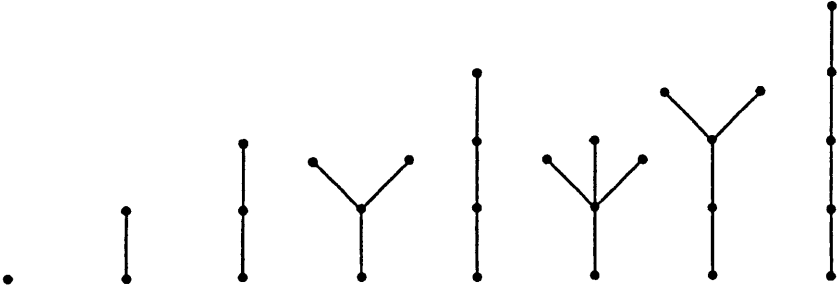
simple embedding: a graph that does not contain edges that cross — see **planar graph**.

spanning tree: a tree in a connected graph which includes every vertex of the graph. In the graph below H is a spanning tree of G .



topology: a branch of mathematics that deals with geometric properties which are unaltered by elastic deformation (e.g. stretching or twisting).

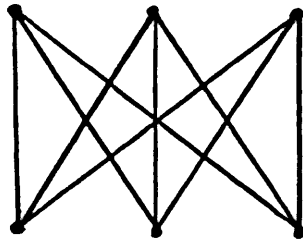
tree: a connected graph that contains no cycle. Trees with up to five vertices are:



triangulation: a plane graph or map in which each region is bounded by three edges.

unavoidable set: a collection of configurations of countries or vertices at least one of which must appear in every map or plane graph.

utilities question: in the plane are three houses and three wells: how can every house be joined to every well by nine paths in all so that no two of these paths cross each other? There is no solution as the following graph indicates:



vertex (of polyhedron): a point in a graph, map or polyhedron where edges meet.

Whitney dual: for any planar graph G , there is a Whitney dual graph G^* constructed as follows: place a vertex of G^* in each region enclosed by edges of G (including the exterior region if G is a finite graph). If two regions enclosed by G have an edge e as a common edge, the vertices of G^* in these regions are joined by an edge e^* in G^* which crosses e . Thus each edge in G is crossed by exactly one edge in G^* . According to this definition, the Whitney dual of a dual graph G^* is the original graph G .

References

1. Robin Wilson, *Four Colours Suffice*, Penguin Books, 2002.
2. N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph Theory, 1736–1936*. Clarendon Press [Oxford], 1998.

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